PROPERTIES OF SOLUTIONS FOR SOME CLASSES OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

BY

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I dedicate my Dissertation work to my family. A special feeling of gratitude to my loving parents, my wife, my sons, my daughter and my sisters.

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TABLE OF CONTENTS

THESIS ABSTRACT

NAME: MOHAMMED DAHAN AHMED QASEM

TITLE OF STUDY: **Properties of Solutions for Some Classes of Nonlinear Fractional Differential Equations**

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In this dissertation we investigate the asymptotic behavior, the decay, the boundedness and the non-existence of solutions of some fractional differential problems.

The present study is motivated by some earlier work on integer order (ordinary) differential equations. It is known that solutions of some differential equations tend to a polynomial (a line for a first order equation) as time goes to infinity. Other solutions oscillate, decay to zero or blow up in finite time.

It is of practical and theoretical importance to extend these results to fractional differential equations due to their numerous applications. Unfortunately, the generalization is not straightforward and many difficulties arise when trying to adopt similar arguments to the ordinary case. Some of the difficulties are in-

herent to the nature of a fractional derivative : it is by definition non-local in time as it involves all the prehistory of the solution (or its derivatives in case of Caputo derivative). Moreover, the kernel involved in this definition is not regular neither summable. These facts do not permit the use of many existing results. In addition to that, it is clear that many properties of integer order derivatives (like the chain rule) are not longer valid for the non-integer order case.

We shall bypass these difficulties by using modified and generalized versions of Gronwall-Bellman inequality, some appropriate estimations of singular terms (like desingularization techniques) and suitable arguments including the test function method (due to Mitidieri and Pohozaev).

We prove convergence to power type functions, power type decay of solutions, boundedness of solutions and non-existence of non-trivial global solutions for two kinds of derivatives: The Riemann-Liouville fractional derivative and the Caputo fractional derivative. Our results are further extended to systems of coupled fractional differential equations.

ملخص بحث

درجة الدكتوراة في الفلسفة

الاســـــــــم: **محمد دحان احمد قاسم**

عنوان البحث: خواص الحلول لبعض الفصائل من المعادلات التفاضلية غير الخطية ذات الرتب غير الصحيحة

التخصــــص: رياضيات

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درسنا في هذة الأطروحة السلوك المقارب، والاضمحلال، والمحدودية وانعدم وجود الحلول لبعض من المعادلات التفاضلية غير الخطيه ذات الرتب غير الصحيحه.

هذة الدراسه مدفوعة ببعض الدرسات السابقة في حالة المعادلات التفاضليه ذات الرتب الصحيحه. من المعروف أن بعض الحلول تنحو الى كثيرة حدود (خط في حالة المعادلة من الرتبة الاولى) عندما يذهب الزمن الى مالانهايه والبعض الاخرى يتذبذب او يضمحل نحو الصفر او ينفجر في وقت محدود.

من المهم من الناحية العملية والنظرية توسيع هذه النتائج الى المعادلات التفاضلية ذات الرتب غير الصحيحة بسبب ظهورها في العديد من التطبيقات. إلا أن هذا التوسيع ليس سهلا إذ تنشأ لدينا العديد من الصعوبات عندما نحاول استخدام الخواص المناظرة للمعادلات التفاضلية ذات الرتب الصحيحة. بعض هذة الصعوبات تكمن في طبيعة المشتقات غير الصحيحة: هي بالتعريف غير موضعية في الزمن إذ تتضمن كل تاريخ الحلول (او مشتقتة في حالة مشتقة كابوتو). علاوة على ذلك فإن نواة التكامل في هذا التعريف غير منتظمة وغير قابلة للتكامل. هذة الحقائق لا تسمح لنا باستخدام النتائج المعروفة حاليا. بالاضافة إلى ذلك فإنه من الواضح أن العديد من الخواص التى تتمتع بها المشتقات ذات الرتب الصحيحه لا يمكن استخدامها في حالة المشتقات ذات الرتب غير الصحيحة مثل قاعدة السلسلة.

سوف نتجاوز هذة الصعوبات باستخدام صيغ معدلة ومطورة من متراجحة قرونوال-بلمان وبعض التقديرات المناسبه للحدود الغير المنتظمة و إستخدم طريقة دالة الاختبار.

نثبت تقارب الحلول الى دوال ذات قوة كسرية موجبة اوسالبة، اودوال محدودة او انعدم وجود حلول عالمية غير تافهة لأشهر نوعين من المشتقات ذات الرتب غير الصحيحة: مشتقة ريمان-ليوفل ومشتقة كابوتو. ايضا قمنا بتوسيع نفس هذة النتائج الى نظام من المعادلات التفاضلية ذات الرتب غير الصحيحة.

CHAPTER 1

INTRODUCTION

1.1 Overview

1.1.1 Fractional differential equations: modeling and significance

Fractional calculus is a new-old subject. The idea of fractional derivative goes back to more than three hundred years. It concerns the generalization of the integer order differentiation and integration to an arbitrary real (or complex) order [1, 2, 3]. Several kinds of fractional derivatives have been defined. They are named after, the Riemann-Liouville, Weyl, Weyl-Liouville, Chen, Bessel, Marchaud, Grünwald-Letnikov, Dzerbashyan, Hadamard, etc...

The Riemann-Liouville derivative is the most commonly used fractional derivative especially by mathematicians. However, the initial conditions associated with this derivative are not always physically meaningful and there are no ways to measure them. Yet, still some of these conditions have physical meaning, see the papers by N. Heymans and I. Podlubny [4] and Podlubny [5].

Physicists and engineers prefer the Caputo derivative which is compatible with the usual initial conditions.

Fractional calculus has been developed by many great mathematicians such as N. H. Abel, G. W. Leibniz, D. S. Laplace, L. Euler, J. Fourier, A. K. Grünwald, J. Hadamard, G. H. Hardy, O. Heaviside, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, H. Weyl, A. Erdelyi, H. Kober, H. Holmgren, A. Zygmund, D. V. Willer,... and nowadays there are hundreds of papers in the subject and several monographs and books.

The first book entirely devoted to the subject is the book by Oldham and Spanier [2]. Other books are: Samko, Kilbas and Marichev [6], Podlubny [3], Kilbas, Strivastava and Trujillo [7] and Mainardi [8].

There are also several journals in the subject: like "Journal of Fractional Calculus", "Fractional Calculus and Applied Analysis"and the recent one "Fractional Dynamic Systems".

The interest in fractional calculus has been accelerated in the past three decades after the publication of the three papers of Bagley and Torvik [9, 10, 11] and the paper by Podlubny [5]. The authors proved by experiments that when using fractional derivatives for viscoelastic materials

- experimental data (amplitude/frequency and dispersions) are in agreement within a broad frequency range,
- time domain responses as stress relaxation and creep are well represented,
- the number of parameters is significantly reduced and
- the fractional model has some other features.

Many events in diverse fields of engineering can be portrayed better and more accurately by differential equations of non-integer order. Namely, they arise naturally in viscoelasticity, porous media, electrochemistry, control, electromagnetic, etc [12, 4].

It has been shown that fractional derivatives and fractional integrals are very efficient in describing for instance anomalous kinetics and continuous time random walks. In general it has been confirmed that fractional derivatives are more suitable and adequate than derivatives of integer orders for the description of properties of many materials (especially materials with memory) and hereditary phenomena and processes. There are many of such systems in atmospheric diffusion of pollution, network traffic, anomalous diffusion, chaotic processes, biology, medicine, modelling and identification, electronics and wave propagation, mechanics, astrophysics, signal processing, chaotic dynamics, optics, powers media [13, 14, 15, 16].

In addition to that, Podlubny established in [5] a geometric interpretation of the fractional integral and a physical interpretation of the fractional derivative.

Fractional derivatives by definition involve all the history of the state through a convolution with a singular kernel. In addition to this singularity, the convolution term is (by definition) non-local in time. This fact complicates considerably the use of the existing methods in the literature.

1.1.2 Significance of the investigation

The study of the long time behavior of solutions of differential problems is in general extremely useful in applications. It has attracted many researchers.

The question of investigating the asymptotic behavior of solutions of general differential problems consists often of determining sufficient conditions ensuring a certain specific (or just exploring the) behavior for large values of time. This task may be simple for simple problems: like for the linear equation $y''(t) = 1/t^2$, $t \geq 1$. This equation admits the general solution $y(t) = \ln t + at + b$, $a, b \in \mathbb{R}$. It is clear that these solutions enjoy the behavior $at + o(t)$ at infinity. However; in the case of nonlinear differential equations we cannot always find exact solutions, for example

$$y'' + (2t)^{-4} y^2 \cos y + (4t)^{-2} (y')^2 \sin^3 y = 0, \ \ t \geq 1.$$

Things become even more complicated when dealing with nonlinear fractional differential equations. Therefore, observing the behavior through the explicit solution is not always possible.

Moreover, there are many other reasons why we study the asymptotic behavior of solutions. We note that existence of asymptotically linear solutions is related, for example, to

1. existence of non-oscillatory solutions,
2. existence of bounded solutions,
3. existence of monotonic solutions,
4. existence of eventually positive (negative) solutions.

Also, sufficient conditions for nonexistence provide necessary conditions for existence of solutions.

1.2 Problem statement

In this thesis we aim to study some properties of solutions for the following general fractional differential equation

$$ {}^{g}D_{0}^{\alpha}y\left(t\right)=f\left[t,y\left(t\right),{}^{g}D_{0}^{\beta}y\left(t\right)\right],\quad 0\leq\beta\leq1,\ \beta<\alpha<2,\ t>0, \tag{1.1} $$

and the system

$$ \begin{cases} {}^{g}D_{0}^{\alpha_1}y_1\left(t\right)=g_1\left(t,{}^{g}D_{0}^{\beta_1}y_1\left(t\right),{}^{g}D_{0}^{\beta_2}y_2\left(t\right)\right),\ t>0,\\ {}^{g}D_{0}^{\alpha_2}y_2\left(t\right)=g_2\left(t,{}^{g}D_{0}^{\beta_1}y_1\left(t\right),{}^{g}D_{0}^{\beta_2}y_2\left(t\right)\right),\ t>0, \end{cases} \tag{1.2} $$

where $0\leq\beta_i\leq1$, $i=1,2$, $\beta_1,\beta_2<\alpha_i<2$, $i=1,2$, and ${}^{g}D_{0}^{\alpha}$ is one of the following derivatives

1. Riemann-Liouville derivative

$$ {}^{RL}D_{0}^{\alpha}y\left(t\right)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}\frac{y(s)}{(t-s)^{\alpha-n+1}}ds,\ t>0. $$

2. Caputo derivative

$$ {}^{C}D_{0}^{\alpha}y\left(t\right)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,\ t\geq0, $$

where $n=-\left[-\alpha\right]$, $[.]$ is the integral part of α.

Equation (1.1) covers many well-known fractional differential equations and ordinary differential equations. For instance, a special case of (1.1) is the Langevin equation when $\alpha = 2$, $\beta = 1$ and $f\left[t, y\left(t\right), y'\left(t\right)\right] = g[t, y\left(t\right)] - \lambda y'\left(t\right)$. Langevin equation

$$my''\left(t\right) = -\lambda y'\left(t\right) + g[t, y\left(t\right)]$$

(as well as some of its fractional versions) is very effective in describing accurately the evolution of some physical phenomena in fluctuating media [17].

1.3 Objectives

We investigate the following issues for solutions of equation (1.1):

1. **Boundedness**: find, or at least prove, the existence of (possibly explicit) bounds for solutions. A priori bounds lead usually to global existence results.

2. **Long time behaviour**: seek sufficient conditions ensuring the convergence of solutions to a certain desirable state. In our case here, the targeted states are, power type functions or the zero state.

3. **Decay and growth**: determine, whenever possible, the rate at which solutions decay to zero, approach a certain attractor or blow up. This is extremely useful in applications.

4. **Non-existence and blow up**: determine (sufficient) conditions ensuring that no (nontrivial) solutions can exist for all time.

Also, we plan to extend some of these results to the system (1.2). This will generalize and extend several existing results from the integer order case to the non-integer order case.

1.4 Methodology

We plan to achieve the objectives through

1. **Considering the corresponding Volterra integral equations**. The original Fractional Differential Problem (FDP) is first transformed to an equivalent Integral Equation (IE) in a suitable underlying space. This (IE) requires, of course, less regularity and encloses the appropriate type of initial data as well as the type of fractional derivative in it.

2. **Applying generalized Gronwall-type inequalities**. The obtained (Volterra) Integral Equation (IE) is then worked out and prepared to fit one of the generalized versions of Gronwall inequality (like Bihari-type inequality). To this end, several suitable estimates, well-known or ad-hoc lemmas, comparison theorems will be used or established. In addition, we are expecting the use of some desingularization techniques to deal with the different singularities appearing in the kernels. This has led to prove new kinds of Gronwall-type inequalities.

3. **L'Hopital rule for fractional derivatives**. We proved L'Hopital rule for fractional derivatives different from the existing basic one in the literature. Indeed, our underlying spaces are set in their upmost generality.

4. **Applying the test function method**. We will adopt an argument based on Mitidieri and Pohozaev technique.

1.5 Literature review

For the problem (1.1) and the system (1.2) we are not aware of many papers on the blow up, boundedness and asymptotic behavior of solutions. In contrast with the integer order case, most of the current publications on related fractional differential equations deal only with the issue of existence and uniqueness.

1.5.1 Asymptotic behavior for integer order problems

In this section, we shall review some works on the asymptotic behavior of solutions of the n-th order nonlinear ordinary differential equation

$$y^{(n)} = f\left(t, y, y', ..., y^{(n-1)}\right). \tag{1.3}$$

We shall be concerned primarily by some sufficient conditions guaranteeing the existence of solutions to this ODE approaching a polynomial of degree $1 \leq m \leq n-1$ as $t \to \infty$. This issue, for ordinary second order differential equation has been studied by many authors.

We may go back in time to 1941 when Caligo [18] proved that, if

$$|A(t)| < \frac{k}{t^{2+r}}, \tag{1.4}$$

for all large t, for some k, $r > 0$, then any solution $y(t)$ of the linear differential equation

$$y''(t) + A(t)\, y(t) = 0, \ \ t > 0, \tag{1.5}$$

can be represented asymptotically as $y(t) = ct + b + o(1)$ when $t \to \infty$, c, $b \in \mathbb{R}$. In [19] (1963), Trench studied the behavior of solutions of the differential equation

$$y'' = (f(t) + g(t))\, y, \ \ t > 0. \tag{1.6}$$

He proved that there is a solution of (1.6) which can be written in the form

$$y = \alpha(t)\, z_1 + \beta(t)\, z_2, \ \ t > 0,$$

with $\lim_{t\to\infty} \alpha(t) = a$ and $\lim_{t\to\infty} \beta(t) = b$, $a, b \in \mathbb{R}$, where z_1 and z_2 are independent solutions of

$$z'' = f(t)\, z, \ \ t > 0,$$

and

$$\int_0^\infty |g(t)|\, y(t)\, dt < \infty.$$

In [20] (1967), Cohen proved that the nonlinear differential equation

$$y''(t) = f(t, y(t)), \ \ t \geq 1, \tag{1.7}$$

has a solution which is asymptotic to $b+ct$ as $t \to \infty$, where b, c are real constants, $c \neq 0$, provided that $f(t,y)$ satisfies the conditions below:

(i) $f(t,y)$ is continuous in $D = \{(t,y) : t \geq 1,\ y \in \mathbb{R}\}$,

(ii) the derivative f_y exists in D and satisfies $f_y(t,y) > 0$ in D,

(iii) $|f(t,y(t))| \leq f_y(t,0)\,|y(t)|$ in D,

(iv) $\int_1^\infty t f_y(t,0)\,dt < \infty$.

The asymptotic behavior of solutions of the nonlinear equation (1.7) has been also studied by Tong [21] (1982), Kusano and Trench [22] (1985) and [23] (1985), Constantin [24] (2005) and others under different conditions.

The differential equation

$$y''(t) = f(t, y(t), y'(t)),\ t \geq 1, \tag{1.8}$$

has been treated by Dannan [25] (1985). He proved that any solution $y(t)$ of (1.8) with initial conditions $y(1) = c_1$ and $y'(1) = c_2$ is asymptotic to $b + ct + o(t)$ as $t \to \infty$, under the following conditions

(i) the function $f(t,u,v)$ is continuous on $D = \{(t,u,v) : t > 1,\ u, v \in \mathbb{R}\}$.

(ii) $|f(t,u,v)| \leq \phi(t)\, g\left(\frac{|u|}{t}\right) + \psi(t)\,|v|$ for $(t,u,v) \in D$, where $\phi(t)$ and $\psi(t)$ are nonnegative continuous functions on $[1,\infty)$.

(iii) $g(u)$ is a nonnegative, continuous, nondecreasing function on $[0, \infty)$, and satisfies

$$|g(\alpha u)| \leq \phi_1(\alpha) g(u),$$

for $\alpha \geq 1$, $u \geq 0$, where $\phi_1(\alpha) > 0$ is continuous for $\alpha > 1$.

(iv) $\int_1^\infty \psi(t)\, dt = k_1 < \infty$, $\int_1^\infty \phi(t)\, dt = k_2 < \infty$ and there exists $K \geq 1$ such that

$$E(t) \int_1^\infty \phi(s) \frac{\phi_1(KE(s))}{E^2(s)} ds \leq K \int_1^\infty \frac{ds}{g(s)},$$

where $E(t) = \exp\left[\int_1^t \psi(s)\, ds\right]$.

Equation (1.8) has been further studied by Constantin [26] (1993), Rogovchenko [27] (1998), Rogovchenko [28] (2000), Mustafa, Rogovchenko [29] (2002), Lipovan [30] (2003) and others.

In [31], Hallam studied the asymptotic behavior of the solutions of the nth order nonhomogeneous differential equation

$$y^{(n)} + f\left(t, y, y', ..., y^{(n-1)}\right) = h(t), \ t \geq 1. \tag{1.9}$$

He proved that there exist solutions $y(t)$ of (1.9) which have the asymptotic behavior

$$\frac{y^{(i)}}{t^{n-i-1}} \sim c_i \,(\neq 0) \in \mathbb{R}, \ i = 0, 1, ..., n-1,$$

under the following two conditions

(i)

$$\left|f\left(t,y,y',...,y^{(n-1)}\right)\right|\le\sum_{i=0}^{n-1}g_i\left(t\right)\left|y^{(i)}\right|^{r_i},\ t\ge t_0\ge 1,$$

where $r_i>0$ and $g_i\left(t\right)$ are continuous, $i=0,1,...,n-1$.

(ii)

$$\int_1^\infty\left|h\left(t\right)\right|dt<\infty.$$

In [32], Philos, Purnaras and Tsamatos considered the n-th order, $n>1$, nonlinear differential equation

$$y^{(n)}=f\left(t,y\right),\ t\ge t_0>0, \tag{1.10}$$

and proved that the differential equation (1.10) has a solution y on the interval $[T,\infty)$ for some $T>0$, which is asymptotic to the polynomial $c_0+c_1t+....+c_mt^m$, for $t\to\infty$, i.e.

$$y\left(t\right)=c_0+c_1t+....+c_mt^m+o\left(1\right)\ \text{ for } t\to\infty.$$

In addition, we have

$$y^{(j)}\left(t\right)=\sum_{i=j}^{m}i\left(i-1\right)...\left(i-j+1\right)c_it^{i-j}+o\left(1\right)\ \text{ for } t\to\infty\ \ \left(j=1,...,m\right),$$

and

$$y^{(k)}\left(t\right)=o\left(1\right)\ \text{ for } t\to\infty,\ k=m+1,...,n-1,$$

provided that $m < n-1$, and

(i) f is a continuous real-valued function on $[t_0, \infty) \times \mathbb{R}$ such that

$$|f(t, y)| \leq p(t)\, g\left(\frac{|y|}{t^m}\right) + q(t) \ \text{ for all } (t, y) \in [t_0, \infty) \times \mathbb{R},$$

where p and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ satisfying

$$\int_{t_0}^{\infty} t^{n-1} p(t)\, dt < \infty \text{ and } \int_{t_0}^{\infty} t^{n-1} q(t)\, dt < \infty,$$

and g is a nonnegative continuous real-valued function on $[0, \infty)$ which is not identically zero.

(ii) There exists a positive constant K so that

$$\left[\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!} p(s)\, ds\right] \sup\left\{g(y) : 0 \leq y \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right\}$$

$$+ \int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!} q(s)\, ds \leq K.$$

1.5.2 Asymptotic behavior for non-integer order problems

In contrast, the problem (1.7) with a fractional derivative in the left hand side has been studied by relatively few researchers. In 2009, Băleanu and Mustafa [33] studied the nonlinear fractional differential problem

$$\begin{cases} {}^{C}D_{0}^{\alpha}y\left(t\right)=f\left(t,y\left(t\right)\right), \ \ 0<\alpha<1, \ t>0 \\ y\left(0\right)=y_{0}, \end{cases} \tag{1.11}$$

where ${}^{C}D$ is the Caputo fractional derivative. They proved that the solution of (1.11) is asymptotic to

$$o\left(t^{b\alpha}\right) \ \text{as } t\rightarrow\infty, \ \ 0<1-b<\alpha,$$

under the conditions

$$\left|f\left(t,y\right)\right|\leq h\left(t\right)g\left(\frac{\left|y\right|}{\left(t+1\right)^{\alpha}}\right), \ t\geq0,$$

and

$$t^{\left(p_{3}/p_{1}\right)\left[1-p_{1}\left(1-\alpha\right)\right]}\left\{\int_{0}^{t}\left[h\left(s\right)\right]^{p_{2}}ds\right\}^{p_{3}/p_{2}}\leq M\left(t+1\right)^{\alpha}, \ t\geq0,$$

for some sufficiently large constant $M, p_1, p_2, p_3 > 1, b \in (0,1), g : [0,\infty) \rightarrow [0,\infty)$ is continuous, nondecreasing function and the function $h : [0,\infty) \rightarrow [0,\infty)$ is continuous and such that

$$t^{\left(p_{3}/p_{1}\right)\left[1-p_{1}\left(1-\alpha\right)\right]}\left\|h\right\|_{L^{p_{2}}(0,t)}^{p_{3}}=O\left(t^{\alpha}\right) \ \text{when } t\rightarrow\infty.$$

In 2010, Băleanu *et al.* [34] considered the linear fractional differential equation

$$\left(D_0^{\alpha} y\right)'(t) + a(t)\, y(t) = 0,\ \ 0 < \alpha < 1,\ \ t > 0. \tag{1.12}$$

They proved that a solution of the equation (1.12) is asymptotic to

$$[b + O(1)]\, t^{\alpha - 1} + [c + o(1)]\, t^{\alpha} \ \text{ as } t \to \infty,$$

where

$$b = \lim_{t \to 0} \left[t^{1-\alpha} y(t)\right] \text{ and } c = \frac{1}{\Gamma(1+\alpha)} \lim_{t \to \infty} D_0^{\alpha} y(t),$$

under the assumptions

(i) $a : (0, \infty) \to \mathbb{R}$ is a continuous function such that

$$\frac{\max\{1, T\}}{\Gamma(1+\alpha)} \int_0^T \frac{|a(s)|}{s^{1-\alpha}} ds + \int_T^{\infty} s^{\alpha} |a(s)|\, ds < 1,\ \ T > 0.$$

(ii)

$$\int_T^{\infty} s^{\alpha+1} |a(s)|\, ds < \infty.$$

Moreover, in 2010, the same authors [35] studied the fractional differential equation

$$D_0^{\alpha}(ty' - y) + a(t)\, y = 0,\ \ \ 0 < \alpha < 1,\ \ t > 0, \tag{1.13}$$

and proved that (1.13) has a solution $y \in C([0,\infty),\mathbb{R}) \cap C^1([0,\infty),\mathbb{R})$, satisfying $\lim_{t\to 0}\left[t^{2-\alpha}y'(t)\right] = 0$ and

$$y(t) = ct + O(t^{\varepsilon}) \text{ as } t \to \infty,\ \varepsilon \in (0,1),\ c \neq 0.$$

In 2011, again the same authors [36] showed that (1.13) has the asymptotic property

$$y(t) = ct + O\left(t^{\alpha-1}\right) \text{ as } t \to \infty,\ c \neq 0,$$

under the assumptions

$$\int_0^{\infty} t\,|a(t)|\,dt + \sup_{t>0} t^{1-\alpha}\int_0^t \frac{s\,|a(s)|}{(t-s)^{1-\alpha}}ds < \infty,$$

and

$$\frac{1}{\Gamma(\alpha)}\left(\int_0^{\infty}\frac{|a(s)|}{s^{1-\alpha}}ds + \chi\right) = k_3 < 1,$$

where

$$\chi = \sup_{t>0} t^{1-\alpha}\int_0^t \frac{|a(s)|}{(t-s)^{1-\alpha}\,s^{1-\alpha}}ds.$$

Furthermore, it is established that the linear fractional equation

$$D_0^{\alpha}(y') + a(t)\,y = 0, \quad 0 < \alpha < 1,\ t > 0, \tag{1.14}$$

has a solution $y \in C([0,\infty),\mathbb{R})$ enjoying the asymptotic property

$$y(t) = b + ct^{\alpha} + O\left(t^{\alpha-1}\right) \text{ as } t \to \infty, \tag{1.15}$$

where

$$b = y(0) \text{ and } c = \frac{1}{\Gamma(1+\alpha)} \lim_{t\to\infty} I_0^{1-\alpha} y',$$

in case

$$\int_T^{\infty} s^{\alpha+1} |a(s)|\, ds < \infty,\ T > 0,$$

and

$$\frac{\max\{1, T^{\alpha}\}}{\Gamma(1+\alpha)} \left[\int_0^T |a(s)|\, ds + \int_T^{\infty} s^{\alpha} |a(s)|\, ds\right] = k < 1.$$

In 2011, Băleanu *et al.* [37] discussed the nonlinear form of (1.14), namely

$$\left(D_0^{\alpha} y'\right)(t) + f(t,y) = 0,\ 0 < \alpha < 1,\ t > 0, \tag{1.16}$$

where the continuous function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(t,y)| \leq F\left(t, \frac{|y|}{(1+t)^{\alpha}}\right),\ t \geq 0,\ y \in \mathbb{R},$$

for some continuous comparison function $F : [0,\infty)^2 \to [0,\infty)$ assumed nondecreasing in the second argument. They proved that solutions of (1.16) have the same asymptotic behavior as (1.15) under the same initial conditions.

In 2012, Medved [38] studied the behavior of solutions of the fractional differential problem with Caputo fractional derivative

$$\begin{cases} {}^{C}D_{a}^{\alpha} y\left(t\right) = f\left(t, y\left(t\right)\right), \quad t \geq a > 1, \; 1 < \alpha < 2, \\ y\left(a\right) = c_0, \; y'\left(a\right) = c_2. \end{cases} \tag{1.17}$$

He proved that problem (1.17) has a solution $y(t)$ which is asymptotic to $b + ct$ as $t \to \infty$, for some b, $c \in \mathbb{R}$, as long as

(i) $f(t, v)$ is continuous in $D = \{(t, v) : t \in [0, \infty), \, v \in \mathbb{R}\}$.

(ii) There are continuous nonnegative functions $h, g : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$, g is nondecreasing and $\gamma > 0$ with $p(\gamma - 1) + 1 > 0$ such that

$$\left|f\left(t, y\right)\right| \leq t^{\gamma - 1} h\left(t\right) g\left(\frac{|y|}{t}\right), \; t > 0, \; \left(t, y\right) \in D,$$

where $p > 1$, $p\left(\alpha - 2\right) + 1 > 0$, $q = \frac{p}{p-1}$, $\gamma = 3 - \alpha + \frac{1}{p}$ and

$$\int_{a}^{\infty} h^{q}\left(s\right) ds < \infty.$$

(iii)

$$\int_{a}^{\infty} \frac{s^{q-1}}{g^{q}\left(s\right)} ds = \infty.$$

One year later, the same author in [39], considered the following generalization

$$
{}^{C}D_{a}^{\alpha+1}y\left(t\right)=f\left(t,y\left(t\right),y'\left(t\right)\right),\ t\geq a>1,\ \alpha\in(0,1), \tag{1.18}
$$

and reached the conclusion that every solution of the equation (1.18) is asymptotic to $b+ct$ as $t\to\infty$, for some b, $c\in\mathbb{R}$, provided that

(i) $f(t,u,v)$ is continuous in $D=\{(t,u,v):t\in[0,\infty),\ u,\ v\in\mathbb{R}\}$.

(ii) There are continuous nonnegative functions $h_i:[0,\infty)\to\mathbb{R}_+$, $i=1,2,3$ and continuous nonnegative and nondecreasing functions $g_j:\mathbb{R}_+\to\mathbb{R}_+$, $j=1,2$ and $\gamma>0$ with $p(\gamma-1)+1>0$ such that

$$
\left|f\left(t,u,v\right)\right|\leq t^{\gamma-1}\left[h_1\left(t\right)g_1\left(\frac{|u|}{t}\right)+h_2\left(t\right)g_2\left(|v|\right)+h_3\left(t\right)\right],\ t>0,\ \ (t,u,v)\in D,
$$

where $p>1$, $p\left(\alpha-1\right)+1>0$, $q=\frac{p}{p-1}$ $\gamma=2-\alpha-\frac{1}{p}$ and

$$
\int_a^\infty h_i^q\left(s\right)ds<\infty,\ i=1,2,3.
$$

(iii)

$$
\int_a^\infty \frac{s^{q-1}}{g_1^q\left(s\right)+g_2^q\left(s\right)}ds=\infty.
$$

In addition to that, he proved that for any solution $y(t)$ of the initial value problem

$$\begin{cases} {}^{C}D_{a}^{\alpha}y\left(t\right)=f\left(t,y\left(t\right)\right),\ t\geq\ a>1,\ \alpha\in\left(n-1,n\right), \\ y\left(a\right)=c_{0},\ y'\left(a\right)=c_{1},\ y^{(n-1)}\left(a\right)=c_{n-1}, \end{cases}$$

defined on the interval $[0,\infty)$ there is a number $c\in\mathbb{R}$ such that

$$y\left(t\right)=\frac{c}{(n-1)!}t^{n-1}+o\left(t^{n-1}\right)\ \text{as}\ t\to\infty,$$

when

(H1) The function $f(t,u)$ is continuous in $D=\left\{(t,u):t\in[a,\infty),\ u\in\mathbb{R}\right\}$.

(H2) There exist continuous nonnegative functions $k_1,\ k_2:[a,\infty)\to\mathbb{R}$, a continuous positive nondecreasing function $g:[0,\infty)\to\mathbb{R}$ and numbers $q>1$, $\gamma>0$ such that

$$k_i=\int_a^\infty k_i^q\left(s\right)ds<\infty,\ i=1,2,$$

$$\left|f\left(t,u\right)\right|\leq t^{\gamma-1}\left[k_1\left(t\right)g_1\left(\frac{|u|}{t^{n-1}}\right)+k_2\left(t\right)\right],\ t>a,\ \ (t,u)\in D.$$

(H3) $p\left(\alpha-1\right)+1>0, p(\gamma-1)+1>0, \gamma=2-\alpha-\frac{1}{p}$, i.e. $\Theta:=p\left(\alpha+\gamma-2\right)+1=0$ where $p=\frac{q}{q-1}$.

(H4)

$$\int_0^\infty\frac{s^{q-1}}{g^q\left(s\right)}ds=\infty.$$

In the paper [40], Brestovanska and Medved considered the fractional initial-value problem

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) + \sum\limits_{i=1}^{m} r_i(t) \int\limits_0^t (t-s)^{\alpha_i - 1} f_i(s, y(s), y'(s))\, ds = 0, \\ y(1) = b_1,\ y'(1) = b_2\ ,\ 0 < \alpha_i < 1,\ i = 1, 2, ..., m,\ t > 0. \end{cases} \tag{1.19}$$

They proved that any solution of (1.19) is asymptotic to a straight line, under the following set of hypotheses

(H1) Every solution of the equation (1.19) is global.

(H2) The functions $f(t, u, v)$ and $f_i(t, u, v)$, $i = 1, 2, ...m$, are continuous on $D = \{(t, u, v) : t \in [0, \infty),\ u, v \in \mathbb{R}\}$ and the functions $r_i(t)$, $i = 1, 2, ...m$, are continuous on $[0, \infty)$.

(H3) There exist continuous functions $h_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, 2, 3$, and continuous, positive and nondecreasing functions $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, u, v)| \leq S e^{-\gamma t} \left[h_1(t)\, g_1\left(\frac{|u|}{t}\right) + h_2(t)\, g_2(|v|) + h_3(t) \right],\ t > 0,$$

where S, $\gamma > 0$.

(H4) There exist continuous, nonnegative functions $h_{ji} : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, 2, 3$; $i = 1, ..., m$, and continuous positive, nondecreasing functions $G_{ji} : \mathbb{R}_+ \to$

$\mathbb{R}_+$, $j = 1, 2$; $i = 1, 2, ..., m$, such that

$$|f_i(t, u, v)| \leq h_{1i}(t)\, G_{1i}\left(\frac{|u|}{t}\right) + h_{2i}(t)\, G_{2i}(|v|) + h_{3i}(t)\,, \ t > 0,$$

for all $(t, u, v) \in D$, $i = 1, 2, .., m$.

(H5) $|r_i(t)| \leq S_i e^{-w_i t}$, $t \geq 0$ where $S_i > 0$, $w_i > 1$, $i = 1, 2, .., m$.

(H6) There exist numbers $p_i > 1$, $i = 1, 2, .., m$, such that $p_i(\alpha_i - 1) + 1 > 0$ with

$$\int_0^\infty h_j^q(s)\, ds < \infty, \quad \int_0^\infty h_{ji}^q(s)\, ds < \infty\ , \ j = 1, 2, 3; \ i = 1, ..., m,$$

where $q = q_1 q_2 ... q_m$, $q_i = \frac{p_i}{p_i - 1}$, $i = 1, 2, ..., m$.

(H7)

$$\int_0^\infty \frac{s^{q-1}}{w(s)} ds = \infty,$$

where

$$w(s) = g_1^q(s) + g_2^q(s) + \sum_{i=1}^m \sum_{j=1}^2 G_{ji}^q(s).$$

In 2015, Medved and Pospíšil [41] studied the fractional initial value problem

$$\begin{cases} {}^C D_a^\alpha y(t) = f\left(t, y(t), {}^C D_a^\beta y(t)\right), \ 0 < \beta < \alpha < 1, \ t \geq a > 0, \\ y(a) = b. \end{cases} \tag{1.20}$$

They proved that any global solution $y(t)$ of the problem (1.20) has the asymptotic property $y(t) = ct^\beta + o\left(t^\beta\right)$ as $t \to \infty$, for some $c \in \mathbb{R}$, provided that

(H1) The function $f(t,u,v)$ is continuous in $D = \{(t,u,v) : t \in [a,\infty),\ u,v \in \mathbb{R}\}$.

(H2) There exist continuous functions $h_1, h_2, h_3, g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$, such that g_1, g_2 are nondecreasing,

$$|f(t,u,v)| \leq t^{\gamma-1}\left[h_3(t) + h_1(t)\, g_1\left(\frac{|u|}{t^\beta}\right) + h_2(t)\, g_2(|v|)\right],\ t \geq a,$$

for some $\gamma \in (1-\frac{1}{p}, 2-\alpha-\beta-\frac{1}{p}]$, $p>1$, $p(\alpha-\beta-1)+1>0$ and

$$H_i = \int_a^\infty h_i^q(s)\, ds < \infty,\ i = 1,2,3,$$

where $q = \frac{p}{p-1}$.

(H3)

$$\int_a^\infty \frac{s^{q-1}}{g_1^q(s) + g_2^q(s)} ds = \infty.$$

1.5.3 Power type decay for problems

The behavior of solutions of the nonlinear problem

$$\begin{cases} D_0^\alpha y(t) = f(t, y(t)),\ 0<\alpha<1,\ t>0, \\ t^{1-\alpha} y(t)\,|_{t=0} = b, \end{cases} \tag{1.21}$$

has been considered by Furati and Tatar in [42]. They proved that solutions decay as a power type function on their interval of existence provided that $f(t,y)$

satisfies the condition

$$|f(t,y)| \leq t^{\mu} e^{-\sigma t} \varphi(t) |y|^{m}, \ \mu \geq 0, \ m > 1, \ \sigma > 0, \ t > 0, \tag{1.22}$$

where $\varphi(t)$ is a continuous function on $\mathbb{R}_{+}$.

In 2012, Furati, Kassim and Tatar [43] studied the nonlinear fractional differential problem

$$\begin{cases} D_{0}^{\alpha,\beta} y(t) = f(t, y(t)), \ 0 < \alpha < 1, \ t > 0, \\ t^{(1-\alpha)(1-\beta)} y(t)|_{t=0} = b, \end{cases} \tag{1.23}$$

where

$$D_{0}^{\alpha,\beta} = I_{0}^{\beta(1-\alpha)} D I_{0}^{(1-\beta)(1-\alpha)},$$

is the Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$. They showed that solutions of (1.23) also decay as a power function under the same condition (1.22) on the function $f(t,y)$.

Finally, we mention that some interesting stability results have been established in [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 42, 54, 55, 56, 57] as well.

1.5.4 Existence and non-existence for problems

The non-existence issue is not well investigated for fractional differential problems in general. In case $\alpha = 1$, $\beta = 0$ and $f(t, y(t)) = y^{m}(t)$ in (1.1) we obtain

$$\begin{cases} y'(t) = y^{m}(t), \ t > 0, \\ y(t)|_{t=0} = b. \end{cases}$$

This problem has, for $m > 1$, the solution

$$y(t) = \left[(1-m)(t+c)\right]^{1/(1-m)}, \ t \geq 0,$$

where

$$c = \frac{b^{1-m}}{1-m}.$$

Observe that, for $m > 1$, the solution blows up in finite time.

When $\alpha = 1$, $\beta = 0$ and $f(t, y(t)) = y^m(t) - y(t)$, the problem (1.1) is equivalent to the Bernoulli differential problem

$$\begin{cases} y'(t) + y(t) = y^m(t), \ t > 0, \\ y(t)\left.\right|_{t=0} = b. \end{cases} \tag{1.24}$$

The solution of (1.24) is given by

$$y(t) = \left[1 + \left(b^{1-m} - 1\right) \exp(m-1)t\right]^{1/(1-m)}, \ t \geq 0.$$

Clearly $y(t)$ blows up in the finite time

$$c = \frac{1}{1-m} \ln\left(1 - b^{1-m}\right), \ m, \ b > 1.$$

In case $0 < \alpha = \beta < 1$ and $f(t, y(t)) = t^{\gamma} |y(t)|^{m}$ in (1.1) we obtain the problem with only one fractional derivative

$$\begin{cases} D_0^{\alpha} y(t) \geq t^{\gamma} |y(t)|^{m}, \ t > 0, \\ I_0^{1-\alpha} y(t) |_{t=0} = b. \end{cases} \tag{1.25}$$

Problem (1.25) has been considered by Laskri and Tatar [58]. It was shown that if $\gamma > -\alpha$ and $1 < m \leq \frac{\gamma+1}{1-\alpha}$, then, Problem (1.25) does not admit global nontrivial solutions when $b \geq 0$.

The existence and uniqueness of solutions for problem (1.1) and system (1.2) have been discussed in [7].

Theorem 1.5.1 *[7] Let $\alpha > 0$ be such that $n - 1 < \alpha \leq n$. Let $l \in \mathbb{N} \backslash \{1\}$ and $\alpha_j \in \mathbb{C}$, $j = 1, ..., l$, be such that*

$$0 = \alpha_0 < \alpha_1 < ... < \alpha_l < \alpha.$$

Let G be an open set in $\mathbb{R}^{l+1}$ and let $f : (a, b] \times G \to \mathbb{R}$ be a function such that $f[t, y, y_1, ..., y_l] \in C_{n-\alpha}[a, b]$ for any $(y, y_1, ..., y_l) \in G$ and satisfies the Lipschitz condition

$$|f[t, y, y_1, ..., y_l] - f[t, Y, Y_1, ..., Y_l]| \leq A_l \sum_{j=0}^{l} |y_j - Y_j|,$$

for all $t \in (a, b]$ and $y, y_1, ..., y_l; Y, Y_1, ..., Y_l \in G$ and where $A_l > 0$ does not depend on $t \in [a, b]$ and let $D_a^{\alpha_j - k_j} y(a) = b_{k_j} \in \mathbb{R}$, $j = 1, ..., n_j$ be fixed numbers, where

$n_j = [\alpha_j] + 1$ for $\alpha_j \notin \mathbb{N}$ and $n_j = \alpha_j$ for $\alpha_j \in \mathbb{N}$. Then, there exists a unique solution y to the Cauchy type problem

$$\begin{cases} D_a^{\alpha} y(t) = f\left[t, y(t), D_a^{\alpha_1} y(t), ..., D_a^{\alpha_l} y(t)\right], \ t > a, \\ D_a^{\alpha-k} y(a) = b_k, \ k = 1, ..., n, \ b_k \in \mathbb{R}, \end{cases}$$

in the space $C_{n-\alpha}^{\alpha}[a,b]$, where

$$C_{n-\alpha}^{\alpha}[a,b] = \{y \in C_{n-\alpha}[a,b] : D_a^{\alpha} y \in C_{n-\alpha}[a,b]\}, \tag{1.26}$$

and

$$C_{n-\alpha}[a,b] = \left\{y : (a,b] \to \mathbb{R} : (t-a)^{n-\alpha} y(t) \in C[a,b]\right\}.$$

Theorem 1.5.2 *[7] Let $j = 1, .., m$, $\alpha_j \in \mathbb{R}$, $n_j - 1 < \alpha_j \leq n_j$, $n_j \in \mathbb{N}$. Let G be an open set in $\mathbb{R}^m$ and let $f_j : (a,b] \times G \to \mathbb{R}$ be functions such that $f_j[t, y_1, ..., y_m] \in C_{n_j - \alpha_j}[a,b]$ for any $(y_1, ..., y_m) \in G$ and satisfy the Lipschitz conditions*

$$|f_k[t, y_1, ..., y_m] - f_k[t, Y_1, ..., Y_m]| \leq \sum_{k=1}^{m} A_k |y_k - Y_k|, \ A_k > 0, \ k = 1, ..., m,$$

for all $t \in (a,b]$ and $y, y_1, ..., y_m; Y, Y_1, ..., Y_m \in G$, and where $A_k > 0$ do not depend on $t \in [a,b]$. Then, there exists a unique solution $(y_1, ..., y_m)$ to the problem

$$\begin{cases} D_a^{\alpha_r} y_r(t) = f_r\left[t, y_1(t), ..., y_m(t)\right], \ r = 1, ..., m, \ t > a, \\ D_a^{\alpha_k - j_k} y_k(a) = b_{j_k}, \ k = 1, ..., m, \ j_k = 1, ..., n_k, \ b_{j_k} \in \mathbb{R} \end{cases}$$

in the space $C_{n-\alpha}^{|\alpha|}[a,b]$, *where*

$$C_{n-\alpha}^{|\alpha|}[a,b]^m = C_{n_1-\alpha_1}^{\alpha_1}[a,b] \times ... \times C_{n_m-\alpha_m}^{\alpha_m}[a,b].$$

This thesis is organized as follows:

In the next chapter, we present the basic definitions, lemmas, properties and notation needed later in this thesis. In Chapter three, the asymptotic behavior of solutions are stated and proved. Chapters four and five are devoted to the decay and nonexistence results, respectively. The same kinds of questions are addressed for systems in Chapters six, seven and eight, respectively.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some definitions, lemmas, properties and notations used in our results. For more details concerning fractional derivatives, we refer the reader to [7, 3, 6].

2.1 Spaces of integrable, absolutely continuous and continuous functions

In this section, we introduce the definition of p-integrable functions, absolutely continuous and weighted space of continuous functions. We also give some characterizations of these modified spaces which will be used later. Moreover, some important embeddings are stated.

Definition 2.1.1 *[7]: Let $0 \leq a < b \leq \infty$. We denote by $L_p(a,b)$, $1 \leq p \leq \infty$, the set of those Lebesgue real-valued measurable functions f on $[a,b]$ for which $\|f\|_p < \infty$, where*

$$\|f\|_p = \left(\int_a^b |f(s)|^p \, ds \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_\infty = ess \sup_{a \leq t \leq b} |f(t)|.$$

Here $ess \sup |f(t)|$ is the essential supremum of the function $|f(t)|$.

Definition 2.1.2 *[7]: Let $\Omega = [a,b]$, $-\infty < a < b < \infty$, and $m \in \mathbb{N}_0 = \{0, 1, ...\}$. We denote by $C^m(\Omega)$ the space of functions f which are m times* *continuously differentiable on Ω with the norm*

$$\|f\|_{\mathbf{C}^m} = \sum_{k=0}^{m} \left\|f^{(k)}\right\|_{\mathbf{C}} = \sum_{k=0}^{m} \max_{t\in\Omega} \left|f^{(k)}(t)\right|, \quad m \in \mathbb{N}_0.$$

In particular, for $m = 0$, $C^0(\Omega) \equiv C(\Omega)$ is the space of continuous function f on Ω with the norm

$$\|f\|_{\mathbf{C}} = \max_{t\in\Omega} |f(t)|.$$

Definition 2.1.3 *[7]: For $n \in \mathbb{N}$, we denote by $AC^n[a,b]$ the space of functions f which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)} \in AC[a,b]$:*

$$AC^n[a,b] = \left\{f : [a,b] \to \mathbb{R} \text{ and } f^{(n-1)} \in AC[a,b]\right\},$$

where $AC[a,b]$ is the space of absolutely continuous functions on $[a,b]$.

Definition 2.1.4 *[7]: We consider the weighted space of continuous functions*

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R}: (t-a)^\gamma f(t) \in C[a,b]\},\ 0 \le \gamma < 1,$$

with the norm

$$\|f\|_{C_\gamma} = \|(t-a)^\gamma f(t)\|_C, \quad C_0[a,b] = C[a,b].$$

Definition 2.1.5 *[7]: For $n \in \mathbb{N}$, we denote by $C_\gamma^n[a,b]$, $0 \le \gamma < 1$, the space of functions f which are continuously differentiable on $[a,b]$ up to order $n-1$ such* *that* $f^{(n)} \in C_\gamma [a,b]$

$$C_\gamma^n [a,b] = \left\{ f \in C^{n-1} [a,b] : f^{(n)} \in C_\gamma [a,b] \right\},$$

with the norm

$$\|f\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \left\| f^{(k)} \right\|_C + \left\| f^{(n)} \right\|_{C_\gamma}, \quad C_\gamma^0 [a,b] = C_\gamma [a,b].$$

From this definition we have the following characterization of the space $C_\gamma^n [a,b]$.

Lemma 2.1.1 ***[7]:*** *Let* $n \in \mathbb{N} = \{1,2,...\}$ *and* $0 \leq \gamma < 1$. *The space* $C_\gamma^n [a,b]$ *consists of those and only those functions* f *which can be represented in the form*

$$f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} \varphi(s)\, ds + \sum\nolimits_{k=0}^{n-1} c_k (t-a)^k, \ t > a,$$

where $\varphi \in C_\gamma [a,b]$ *and* c_k, $k = 0,1,...,n-1$, *are arbitrary constants.*

Moreover,

$$\varphi(t) = f^{(n)}(t), \qquad c_k = \frac{f^{(k)}(a)}{k!}, \qquad k = 0,1,...,n-1, \ t > a.$$

Definition 2.1.6 ***[7]:*** *The Euler Gamma function* $\Gamma(z)$ *is defined by the so–called Euler integral of the second kind:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ z > 0,$$

where $t^{z-1} = e^{(z-1)\log t}$.

2.2 Riemann-Liouville fractional integrals and fractional derivatives

In this section, we introduce the definitions of the Riemann-Liouville fractional integrals, fractional derivatives and present some of their properties in the space of continuous functions.

Definition 2.2.1 *[7]: The Riemann-Liouville left-sided fractional integral* $I_a^\alpha f$ *of order* $\alpha > 0$ *is defined by*

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ t > a,$$

provided that the integral exists. Here $\Gamma(\alpha)$ *is the Gamma function. When* $\alpha = 0$*, we set*

$$I_a^0 f = f.$$

Definition 2.2.2 *[7]: The Riemann-Liouville right-sided fractional integral* $I_{b^-}^\alpha f$ *of order* $\alpha > 0$ *is defined by*

$$I_{b^-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \ t < b,$$

provided that the integral exists. When $\alpha = 0$, we define

$$I_{b-}^{0} f = f.$$

Definition 2.2.3 *[7]: The Riemann-Liouville left-sided fractional derivative $D_a^\alpha f$ of order $\alpha \geq 0$, $n-1 \leq \alpha < n$, $n = -[-\alpha]$, is defined by*

$$D_a^\alpha f(t) = \left(\frac{d}{dt}\right)^n I_a^{n-\alpha} f(t),\ t > a,$$

that is

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,\ t > a.$$

In particular, when $\alpha = n$ we have $D_a^\alpha f = D^n f$, and when $\alpha = 0$, $D_a^0 f = f$.

Definition 2.2.4 ***[7]:*** *The Riemann-Liouville right-sided fractional derivative $D_{b-}^\alpha f$ of order $\alpha \geq 0$, $n-1 \leq \alpha < n$, $n = -[-\alpha]$, is defined by*

$$D_{b-}^\alpha f(t) = \left(-\frac{d}{dt}\right)^n I_{b-}^{n-\alpha} f(t),\ t < b,$$

that is

$$D_{b-}^\alpha f(t) = -\frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(s)}{(s-t)^{\alpha-n+1}} ds,\ t < b.$$

In particular, when $\alpha = n$ we have $D_{b-}^\alpha f = (-1)^n D^n f$, and when $\alpha = 0$, $D_{b-}^0 f = f$.

It can be directly verified that the Riemann-Liouville fractional integral (and fractional derivative) of the power function $(t-a)^{\beta-1}$, $\beta > 0$, yield the same power function with α added (or subtracted) from the power β with a certain coefficient in front of this power function.

Property 2.2.1 *[7]: If $\alpha \geq 0$ and $\beta > 0$, then*

$$I_a^{\alpha}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1} \quad , \quad \alpha > 0, \; t > a,$$

$$I_{b^-}^{\alpha}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1} \quad , \quad \alpha > 0, \; t < b,$$

$$D_a^{\alpha}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1} \quad , \quad \alpha \geq 0, \; t > a,$$

$$D_{b^-}^{\alpha}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1} \quad , \quad \alpha \geq 0, \; t < b.$$

In particular, if $\beta = 1$ and $\alpha \geq 0$, then the Riemann-Liouville fractional derivative of a constant is not equal to zero:

$$(D_a^{\alpha}1)(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \; t > a, \; \textit{and} \; (D_{b^-}^{\alpha}1)(t) = \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)}, \; 0 < \alpha < 1, \; t < b.$$

On the other hand, for $i = 1, 2, ..., -[-\alpha]$,

$$D_a^{\alpha}(t-a)^{\alpha-i} = 0 \; \textit{and} \; D_{b^-}^{\alpha}(b-t)^{\alpha-i} = 0.$$

The existence of the fractional derivatives D_a^{α} and $D_{b^-}^{\alpha}$ in the space $C_{\gamma}^{n}[a,b]$ is guaranteed by the following lemma.

Lemma 2.2.1 *[7]: Let $\alpha \geq 0$, $n = -[-\alpha]$ and $0 \leq \gamma < 1$. If $f \in C_{\gamma}^{n}[a,b]$, then* *the fractional derivatives* D_a^α *and* $D_{b^-}^\alpha$ *exist on* $(a,b]$ *and* $[a,b)$ *respectively, and can be represented in the forms*

$$D_a^\alpha f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)} (t-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)\,ds}{(t-s)^{\alpha-n+1}},\ t>a,$$

and

$$D_{b^-}^\alpha f(t) = \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(1+k-\alpha)} (b-t)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(s)\,ds}{(s-t)^{\alpha-n+1}},\ t<b,$$

respectively.

The Riemann-Liouville fractional integral (Definition 2.2.1) satisfies the following semigroup property.

Lemma 2.2.2 *[7]: Let* $\alpha>0$, $\beta>0$ *and* $0\le\gamma<1$. *If* $f\in C_\gamma[a,b]$, *then*

$$I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t),\ t\in(a,b].$$

The following assertion shows that the fractional differentiation is an inverse operation to the fractional integration from the left.

Lemma 2.2.3 ***[7]:*** *Let* $\alpha>0$ *and* $0\le\gamma<1$. *If* $f\in C_\gamma[a,b]$, *then*

$$D_a^\alpha I_a^\alpha f(t) = f(t),\ t\in(a,b].$$

Another composition property between the fractional differentiation operator

(Definition 2.2.3) and the fractional integration (Definition 2.2.1) is given next.

Property 2.2.2 *[7]: Let $\alpha > \beta > 0$ and $0 \leq \gamma < 1$. If $f \in C_\gamma [a, b]$, then*

$$D_a^\beta I_a^\alpha f(t) = I_a^{\alpha-\beta} f(t), \ t \in (a, b].$$

In particular, when $\beta = k \in \mathbb{N}$ and $\alpha > k$, then $D_a^k I_a^\alpha f = I_a^{\alpha-k} f$.

Property 2.2.3 *[7]: Let $\alpha \geq 0$, $m \in \mathbb{N}$ and $D = d/dt$.*

(a) If the fractional derivatives D_a^α and $D_a^{\alpha+m}$ exist, then

$$D^m D_a^\alpha = D_a^{\alpha+m}.$$

(b) If the fractional derivatives D_{b-}^α and $D_{b-}^{\alpha+m}$ exist, then

$$D^m D_{b-}^\alpha = (-1)^m D_{b-}^{\alpha+m}.$$

Now we consider some other properties of the Riemann-Liouville fractional integral (Definition 2.2.1) in the space $C_\gamma [a, b]$ defined in Definition 2.1.4. The existence of the fractional integral I_a^α in the space $C_\gamma [a, b]$ is given by the following lemma.

Lemma 2.2.4 *[7]: The following hold*

(a) Let $\alpha > 0$ and $0 \leq \gamma < 1$.

If $\gamma > \alpha$, then the fractional integration operator I_a^α is bounded from $C_\gamma [a, b]$

into $C_{\gamma-\alpha}[a,b]$

$$\|I_a^\alpha f\|_{C_{\gamma-\alpha}} \leq k_1 \|f\|_{C_\gamma},\ k_1 = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular I_a^α *is bounded in* $C_\gamma[a,b]$.

(b) If $\gamma \leq \alpha$, *then the fractional integration operator* I_a^α *is bounded from* $C_\gamma[a,b]$ *into* $C[a,b]$

$$\|I_a^\alpha f\|_C \leq k_2 \|f\|_{C_\gamma},\ k_2 = (b-a)^{\alpha-\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular I_a^α *is bounded in* $C_\gamma[a,b]$.

The following result provides another composition of the fractional integration operator I_a^α with the fractional differentiation operator D_a^α.

Lemma 2.2.5 *[7]: Let* $\alpha > 0$, $0 \leq \gamma < 1$, $n = -[-\alpha]$. *If* $f \in C_\gamma[a,b]$ *and* $I_a^{n-\alpha} f \in C_\gamma^n[a,b]$, *then*

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \sum_{i=1}^{n} \frac{\left(D^{n-i} I_a^{n-\alpha} f\right)(a)}{\Gamma(\alpha-i+1)} (t-a)^{\alpha-i}, \quad t > a.$$

In particular, if $0 < \alpha < 1$ *then*

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1},\ t > a.$$

Furthermore, we have the following important lemmas.

Lemma 2.2.6 ***[59]:*** *If* $0 \le \gamma < 1$, *then the fractional integration operator* I_a^α *of order* α *is bounded in* $C_\gamma [a, b]$

$$\|I_a^\alpha g\|_{C_\gamma} \le \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} (b-a)^\alpha \|g\|_{C_\gamma},$$

here I_a^α *is the Riemann-Liouville fractional integral operator and* $g \in C_\gamma [a, b]$.

Lemma 2.2.7 ***[59]:*** *The fractional integration operator* I_a^α *of order* α *is a mapping from* $C[a, b]$ *to* $C[a, b]$, *and*

$$\|I_a^\alpha g\|_C \le \frac{(b-a)^\alpha}{\alpha\Gamma(\alpha)} \|g\|_C,$$

where $g \in C[a, b]$.

2.3 Caputo fractional derivative

In this section, we present the definitions and some properties of the Caputo fractional derivatives which will be involved in the problems investigated later.

Definition 2.3.1 *[7]: For* $\alpha \ge 0$, *the Caputo left-sided and right-sided fractional derivatives of order* α, $n = -[-\alpha]$ *are defined by*

$${}^C D_a^\alpha f(t) = D_a^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right], \ t > a$$

and

$$^{C}D_{b^-}^{\alpha} f(t) = D_{b^-}^{\alpha} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (b-t)^k \right], \ t < b,$$

respectively.

Definition 2.3.2 *[7]: Let $\alpha > 0$, $n = -[-\alpha]$. If $f \in AC^n[a,b]$, then $^{C}D_a^{\alpha} f$ and $^{C}D_{b^-}^{\alpha} f$ exist almost everywhere on $[a,b]$ and are represented by*

$$^{C}D_a^{\alpha} f = I_a^{n-\alpha} D^n f \text{ and } ^{C}D_{b^-}^{\alpha} f = (-1)^n I_{b^-}^{n-\alpha} D^n f.$$

The following result provides a formula for the composition of the fractional differentiation operator $^{C}D_a^{\alpha}$ with the fractional integration operator I_a^{α}. It shows that fractional differentiation is not the right inverse operator of the fractional integral in general.

Lemma 2.3.1 *[7]: Let $\alpha > 0$, $n = -[-\alpha]$. If $f \in AC^n[a,b]$ or $f \in C^n[a,b]$, then*

$$I_a^{\alpha}\ ^{C}D_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k, \ t > a.$$

In particular, if $0 < \alpha \leq 1$ and $f \in AC[a,b]$ or $f \in C[a,b]$, then

$$I_a^{\alpha}\ ^{C}D_a^{\alpha} f(t) = f(t) - f(a), \ t > a.$$

2.4 Some important results

In this section, we present some other important definitions, lemmas, theorems and properties. These will determine the assumptions, tools and the methods utilized in our results later.

Theorem 2.4.1 ***([60], Bihari inequality)***: *Let u and f be nonnegative continuous functions defined on $\mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$). Let $w(u)$ be a continuous nondecreasing function defined on $\mathbb{R}_+$ and $w(u) > 0$ on $(0,\infty)$. If*

$$u\left(t\right) \leq k + \int_0^t f\left(s\right) w\left(u\left(s\right)\right) ds,$$

for $t \in \mathbb{R}_+$, where k is a nonnegative constant, then for $0 \leq t \leq t_1$,

$$u\left(t\right) \leq G^{-1}\left(G\left(k\right) + \int_0^t f\left(s\right) ds\right),$$

where

$$G\left(r\right) = \int_{r_0}^r \frac{ds}{w\left(s\right)}, \ \ r > 0, \ \ r_0 > 0,$$

and G^{-1} is the inverse function of G and $t_1 \in \mathbb{R}_+$ is chosen so that

$$G\left(k\right) + \int_0^t f\left(s\right) ds \in Dom\left(G^{-1}\right),$$

for all $t \in \mathbb{R}_+$ lying in the interval $0 \leq t \leq t_1$.

When w is the identity we get the following theorem.

Theorem 2.4.2 ***([60], Gronwall-Bellman inequality):*** *Let u and f be continuous and nonnegative functions defined on $\mathbb{R}_+$, and let c be a nonnegative constant. Then, the inequality*

$$u(t) \leq c + \int_0^t f(s)\, u(s)\, ds, \ \ t \geq 0,$$

implies that

$$u(t) \leq c \exp\left(\int_0^t f(s)\, ds\right), \ \ t \geq 0.$$

Theorem 2.4.3 ***[60]:*** *Let u and f be continuous and nonnegative functions defined on $\mathbb{R}_+$, and let $n(t)$ be a continuous, positive and nondecreasing function defined on $\mathbb{R}_+$; then*

$$u(t) \leq n(t) + \int_0^t f(s)\, u(s)\, ds, \ \ t \geq 0,$$

implies that

$$u(t) \leq n(t) \exp\left(\int_0^t f(s)\, ds\right), \ \ t \geq 0.$$

Lemma 2.4.1 *(**[7]**, Fractional Integration by Parts): Let $\alpha > 0$, $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then*

$$\int_a^b \varphi(t)\, I_a^\alpha \psi(t)\, dt = \int_a^b \psi(t)\, I_{b-}^\alpha \varphi(t)\, dt.$$

Lemma 2.4.2 ***[61]***: *Let $\alpha > 0$ and $n = -[-\alpha]$. If $f,\ I_{b-}^{n-\alpha} g \in AC^n [a,b]$, then*

$$\int_a^b g(t)\,{}^C D_{a+}^{\alpha} f(t)\,dt = \int_a^b f(t)\, D_{b-}^{\alpha} g(t)\,dt + \sum_{i=0}^{n-1} \left[f^{(i)}(t)\, D_{b-}^{\alpha-i-1} g(t) \right]_{t=a}^{b}.$$

Lemma 2.4.3 ***[62]:*** *If $\lambda,\ \nu,\ \omega > 0$, then for any $t > 0$, we have*

$$\int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \leq C t^{\nu-1},$$

where C is a positive constant independent of t. In fact,

$$C = \max\left\{1, 2^{1-\nu}\right\} \Gamma(\lambda) \left(1 + \lambda(\lambda+1)/\nu\right) \omega^{-\lambda}.$$

Lemma 2.4.4 ***[63, 64]:*** *We have, for positive a, b, the inequalities*

$$a^r + b^r \leq (a+b)^r \leq 2^{r-1}(a^r + b^r),\ \ r \geq 1,$$

and

$$2^{r-1}(a^r + b^r) \leq (a+b)^r \leq a^r + b^r, \quad 0 \leq r \leq 1.$$

CHAPTER 3

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR FRACTIONAL DIFFERENTIAL PROBLEMS OF ORDER BETWEEN ONE AND TWO

This chapter is devoted to studying the asymptotic behavior of solutions for the following general fractional differential equation

$$D_0^{\alpha}y\left(t\right)=f\left[t,y\left(t\right),D_0^{\beta}y\left(t\right)\right],\qquad 0\leq\beta\leq1,\ \beta<\alpha<2,\ t>0, \tag{3.1}$$

in a weighted space of continuous functions, where D_0^{σ} is either the Riemann-Liouville derivative or the Caputo derivative.

We shall establish some conditions ensuring power functions asymptotic behavior. These results can be proved with the help of the Gronwall-Bellman inequality.

3.1 Preliminaries

In this section, we prove some results regarding the asymptotic behavior of fractional integrals.

Lemma 3.1.1 *Let g be a real valued function defined on $[a,b]$. Let $n\in\mathbb{N}$ and $0\leq\gamma<1$. Then, $g^{(n)}\in C_{\gamma}\left[a,b\right]$ if and only if $g\in C_{\gamma}^{n}\left[a,b\right]$.*

Proof. First, we prove the necessity. Let $g^{(n)}\in C_{\gamma}\left[a,b\right]$ and we want to prove that $g\in C_{\gamma}^{n}\left[a,b\right]$. Given $\varepsilon>0$, then $g^{(n)}\in C\left[a+\varepsilon,b\right]$ and thus, by the Fundamental Theorem of Calculus, we get

$$g\left(t\right)=\sum_{k=0}^{n-1}\frac{g^{(k)}\left(a+\varepsilon\right)}{k!}\left(t-a-\varepsilon\right)^{k}+I_{a+\varepsilon}^{n}D^{n}g\left(t\right),\ t\in\left[a+\varepsilon,b\right]. \tag{3.2}$$

Since $g^{(n)} \in C_\gamma[a,b] \subset L_1(a,b)$, then $I_a^1 g^{(n)}(t)$ is bounded on $[a,b]$ and

$$\left|I_a^1 g^{(n)}(t) - I_{a+\varepsilon}^1 g^{(n)}(t)\right| \leq \int_a^{a+\varepsilon} (s-a)^{-\gamma} \left|(s-a)^\gamma g^{(n)}(s)\right| ds$$

$$\leq M \int_a^{a+\varepsilon} (s-a)^{-\gamma} ds = \frac{M}{1-\gamma} \varepsilon^{1-\gamma} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore

$$\lim_{\varepsilon \to 0} I_{a+\varepsilon}^1 g^{(n)}(t) = I_a^1 g^{(n)}(t), \ t \in [a,b].$$

Thus, by taking the limit of (3.2), we obtain

$$g(t) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a^+)}{k!} (t-a)^k + I_a^n D^n g(t), \ t > a.$$

Now, clearly $g^{(k)}(a^+)$, $k = 0, ..., n-1$, are finite and the result follows from Lemma 2.1.1. The other direction follows directly from the definition of $C_\gamma^n[a,b]$. ∎

Remark 3.1.1 *Note that* $C_\gamma^n[a,b] \subset AC^n[a,b]$, $n \geq 1$.

Lemma 3.1.2 *Let* $0 < \alpha < 1$ *and* $0 \leq \gamma < 1$. *If* $f \in C_\gamma[a,b]$ *and* $I_a^{1-\alpha} f \in C_\gamma^1[a,b]$, *then for* $0 \leq \beta \leq \alpha < 1$ *we have*

$$D_a^\beta f(t) = I_a^{\alpha-\beta} D_a^\alpha f(t) + \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha-\beta)} (t-a)^{\alpha-\beta-1}, \ t \in (a,b].$$

Proof. Note that $f \in C_\gamma[a,b]$ *and* $I_a^{1-\alpha} f \in C_\gamma^1[a,b]$, then by virtue of Lemma

2.2.5 we conclude

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}, \ t \in (a,b]. \tag{3.3}$$

Also since $I_a^{1-\alpha} f \in C_\gamma^1 [a,b]$ then $D_a^\alpha f = D I_a^{1-\alpha} f \in C_\gamma [a,b]$, see Definition 2.1.5. Applying D_a^β to both sides of (3.3), using Properties 2.2.1 and 2.2.2, we find

$$D_a^\beta f(t) = I_a^{\alpha-\beta} D_a^\alpha f(t) + \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha-\beta)} (t-a)^{\alpha-\beta-1}, \ t \in (a,b].$$

∎

Lemma 3.1.3 *Let $0 < \beta \leq \alpha < 1$. If $f \in AC[a,b]$, then*

$$^C D_0^\beta f = I_0^{\alpha-\beta} \ ^C D_0^\alpha f.$$

Proof. The result follows directly from the semigroup property, Lemma 2.2.2. In fact

$$^C D_0^\beta f = I_0^{1-\beta} f' = I_0^{\alpha-\beta} I_0^{1-\alpha} f' = I_0^{\alpha-\beta} \ ^C D_0^\alpha f.$$

∎

The following lemma describes the asymptotic behavior of the Riemann-Liouville fractional integral of a summable function.

Lemma 3.1.4 *Let $f \in L_1(0,\infty)$, then*

$$\lim_{t\to\infty} \frac{1}{t^\alpha} I_0^{\alpha+1} f(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(s)\, ds = \frac{1}{\Gamma(\alpha+1)} I_0^1 f(\infty), \ \alpha > 0.$$

Proof. Indeed, in view of the Definition 2.2.1, we see that

$$\left|\frac{1}{t^{\alpha}} I_0^{\alpha+1} f(t) - \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} f(s)\, ds\right|$$

$$\begin{aligned}
&= \left|\frac{1}{\Gamma(\alpha+1)} \int_0^t \frac{(t-s)^{\alpha}}{t^{\alpha}} f(s)\, ds - \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} f(s)\, ds\right| \\
&= \frac{1}{\Gamma(\alpha+1)} \left|\int_0^t \left(1-\frac{s}{t}\right)^{\alpha} f(s)\, ds - \int_0^{\infty} f(s)\, ds\right| \\
&= \frac{1}{\Gamma(\alpha+1)} \left|\int_0^{\infty} \chi_{[0,t]} \left(1-\frac{s}{t}\right)^{\alpha} f(s)\, ds - \int_0^{\infty} f(s)\, ds\right| \\
&= \frac{1}{\Gamma(\alpha+1)} \left|\int_0^{\infty} \left[\chi_{[0,t]} \left(1-\frac{s}{t}\right)^{\alpha} - 1\right] f(s)\, ds\right| \\
&\le \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} \left|\chi_{[0,t]} \left(1-\frac{s}{t}\right)^{\alpha} - 1\right| |f(s)|\, ds,\ t>0,
\end{aligned}$$

where

$$\chi_{[0,t]}(s) = \begin{cases} 1,\ s \in [0,t],\ t>0, \\ 0,\ s \notin [0,t],\ t>0. \end{cases}$$

It is clear that

$$\lim_{t\to\infty} \chi_{[0,t]} \left(1-\frac{s}{t}\right)^{\alpha} = 1.$$

Using the Dominated Convergence Theorem (continuous version) [65], we obtain

$$\lim_{t\to\infty} \int_0^{\infty} \left|\chi_{[0,t]} \left(1-\frac{s}{t}\right)^{\alpha} - 1\right| |f(s)|\, ds = \int_0^{\infty} \lim_{t\to\infty} \left|\chi_{[0,t]} \left(1-\frac{s}{t}\right)^{\alpha} - 1\right| |f(s)|\, ds = 0.$$

∎

Now, we prove L'Hopital rule for Riemann-Liouville and Caputo fractional derivatives, respectively.

Lemma 3.1.5 *Let $0 < \alpha < 1$ and $0 \le \gamma < 1$. Assume that $y \in C_\gamma[0,\infty)$ and $I_0^{1-\alpha} y \in C_\gamma^2[0,\infty)$. Then*

$$\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \lim_{t\to\infty} \frac{D_0^\alpha y(t)}{\Gamma(\alpha+1)}. \tag{3.4}$$

Proof. In view of Lemma 2.2.5, with α replaced by $1+\alpha$ and $n = 2$, we get

$$I_0^{1+\alpha} D_0^{1+\alpha} y(t) = y(t) - \frac{\left(I_0^{1-\alpha} y\right)(0)}{\Gamma(\alpha)} t^{\alpha-1} - \frac{D_0^\alpha y(0)}{\Gamma(1+\alpha)} t^\alpha, \ t > 0. \tag{3.5}$$

Dividing both sides of (3.5) by t^α and taking the limit as $t \to \infty$, we obtain

$$\begin{aligned} \lim_{t\to\infty} \frac{y(t)}{t^\alpha} &= \frac{D_0^\alpha y(0)}{\Gamma(1+\alpha)} + \lim_{t\to\infty} \frac{1}{t^\alpha} I_0^{1+\alpha} D_0^{1+\alpha} y(t) \\ &= \frac{D_0^\alpha y(0)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha+1)} \lim_{t\to\infty} I_0^1 D_0^{1+\alpha} y(t), \end{aligned} \tag{3.6}$$

where we have used Lemma 3.1.4. On the other hand, we find

$$I_0^1 D_0^{1+\alpha} y(t) = D_0^\alpha y(t) - D_0^\alpha y(0), \ t > 0, \tag{3.7}$$

and (3.4) follows directly from (3.6) and (3.7). ∎

Lemma 3.1.6 *Let $0 < \alpha < 1$ and $0 \le \gamma < 1$. Assume that $y \in AC[0,\infty)$ and $I_0^{1-\alpha} y' \in C_\gamma^1[0,\infty)$. Then*

$$\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \lim_{t\to\infty} \frac{{}^C D_0^\alpha y(t)}{\Gamma(1+\alpha)}. \tag{3.8}$$

Proof. Since $y \in AC[0,\infty) \subset C_\gamma[0,\infty)$ *and* $I_0^{1-\alpha}y' \in C_\gamma^1[0,\infty)$, we can apply Lemma 2.2.5 to get

$$I_0^\alpha D_0^\alpha y'(t) = y'(t) - \frac{I_0^{1-\alpha}y'(0)}{\Gamma(\alpha)}t^{\alpha-1},\ t > 0. \tag{3.9}$$

Applying I_0^1 to both sides of (3.9), using Property 2.2.1 (with $\beta = \alpha$) and Lemma 2.2.2, we arrive at

$$y(t) = y(0) + \frac{I_0^{1-\alpha}y'(0)}{\Gamma(\alpha+1)}t^\alpha + I_0^{1+\alpha}D_0^\alpha y'(t),\ t > 0. \tag{3.10}$$

Dividing both sides of (3.10) by t^α and taking the limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty}\frac{y(t)}{t^\alpha} = \frac{I_0^{1-\alpha}y'(0)}{\Gamma(1+\alpha)} + \lim_{t\to\infty}\frac{1}{t^\alpha}I_0^{1+\alpha}D_0^\alpha y'(t)$$

$$= \frac{I_0^{1-\alpha}y'(0)}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)}\lim_{t\to\infty} I_0^1 D_0^\alpha y'(t), \tag{3.11}$$

where we have used Lemma 3.1.4. On the other hand, we deduce

$$I_0^1 D_0^\alpha y'(t) = I_0^1 D I_0^{1-\alpha}y'(t) = I_0^{1-\alpha}y'(t) - I_0^{1-\alpha}y'(0) = {}^C D_0^\alpha y(t) - {}^C D_0^\alpha y(0),\ t > 0, \tag{3.12}$$

and (3.8) follows directly from (3.11) and (3.12).

∎

3.2 Some useful inequalities

In this section, we establish inequalities that involve special classes of functions. These inequalities will be used to obtain our main results.

We begin by defining the following classes of functions:

$$\mathbf{\Phi} = \{\varphi \in C(0,\infty) : \varphi \text{ is positive and nondecreasing on } (0,\infty),$$

$$\frac{1}{u}\varphi(v) \leq \varphi\left(\frac{v}{u}\right), \; v>0, \, u \geq 1\}. \tag{3.13}$$

$$\mathbf{H}_k = \left\{h \in L_1(0,\infty), \; h \text{ is positive and } s^k h \in L_1(1,\infty), \; k > -1\right\}. \tag{3.14}$$

$$\mathbf{\Psi} = \{F : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } 0 \leq F(t,u) - F(t,v) \leq N(t)(u-v),$$

$$t > 0 \text{ and } u \geq v \geq 0, \text{ for some continuous function } N \text{ on } \mathbb{R}_+\}. \tag{3.15}$$

The class of functions $\mathbf{\Phi}$ has been widely employed in the literature, see for instance [66]. Three simple functions belonging to $\mathbf{\Phi}$, $\mathbf{H}_k$ and $\mathbf{\Psi}$ are

$$\varphi(t) = \sum_{i=1}^{n} t^{\alpha_i}, \; \alpha_i \leq 1, \; i = 1,2,...,n, \; t>0,$$

$$h(t) = e^{-t} \text{ and } F(t,u) = ue^t, \; t>0, \; u \geq 0,$$

respectively.

Remark 3.2.1 *If* $\varphi \in \mathbf{\Phi}$, *then* $\int_{x_0}^{\infty} \frac{ds}{\varphi(s)} = \infty$, $x>0$, *by taking* $v=u$, $\frac{1}{u}\varphi(u) \leq \varphi(1)$.

In what follows we give some properties and inequalities that involve these classes of functions. Clearly we have

Lemma 3.2.1 *The spaces* $\mathbf{\Phi}$, $\mathbf{H}_k$ *and* $\mathbf{\Psi}$ *are closed under addition and scalar multiplication.*

Lemma 3.2.2 *Let* $z(t)$ *and* $h(t)$ *be continuous and nonnegative functions defined for* $t \geq 0$, $\varphi \in \mathbf{\Phi}$ *and* $c_i \in \mathbb{R}$, $i = 1, 2, 3$. *Then*

$$z(t) \leq c_1 + c_2 t^\gamma + c_3 t^\gamma \int_0^t h(s)\varphi(z(s))\, ds, \quad \gamma,\ t \geq 0, \tag{3.16}$$

implies

$$z(t) \leq \begin{cases} G^{-1}\left(G(|c_1| + |c_2|) + |c_3| \int_0^t h(s)ds\right),\ 0 \leq t < 1 \\ t^\gamma G^{-1}\left(G(A) + |c_3| \int_1^t s^\gamma h(s)ds\right),\ t \geq 1, \end{cases} \tag{3.17}$$

where G^{-1} *is the inverse function of* $G(x) = \int_{x_0}^x \frac{ds}{\varphi(s)}$,

$$A = |c_1| + |c_2| + |c_3|\, \varphi\left(G^{-1}(K)\right) \int_0^1 h(s)ds,$$

and

$$K = G(|c_1| + |c_2|) + |c_3| \int_0^1 h(s)ds < \infty.$$

Proof. We begin by noting that from the definition of $\mathbf{\Phi}$ the functions G and G^{-1} are increasing, continuous and defined on $(0, \infty)$ and $(G(0^+), \infty)$, respectively.

For $0 \le t < 1$ we have from (3.16)

$$z(t) \le |c_1| + |c_2| + |c_3| \int_0^t h(s)\varphi(z(s))\,ds,$$

and the first inequality of (3.17) follows directly from Bihari's inequality (Theorem 2.4.1).

For $t \ge 1$ we have from (3.16) the estimate

$$\frac{z(t)}{t^\gamma} \le |c_1| + |c_2| + |c_3| \int_0^t h(s)\varphi(z(s))\,ds$$

$$\le |c_1| + |c_2| + |c_3| \int_0^1 h(s)\varphi(z(s))\,ds + |c_3| \int_1^t h(s)\varphi(z(s))\,ds. \tag{3.18}$$

Therefore, from the first inequality of (3.17) and (3.18) we have

$$\frac{z(t)}{t} \le A + |c_3| \int_1^t h(s)\varphi(z(s))\,ds,\ \ t \ge 1. \tag{3.19}$$

In view of the definition of $\mathbf{\Phi}$, we can rewrite (3.19) in the form

$$\frac{z(t)}{t^\gamma} \le A + |c_3| \int_1^t s^\gamma h(s)\varphi\left(\frac{z(s)}{s^\gamma}\right) ds,\ \ t \ge 1. \tag{3.20}$$

Now, the second inequality of (3.17) follows immediately from Bihari's inequality (Theorem 2.4.1). ∎

When φ is the identity we get the following corollary.

Corollary 3.2.1 *Let $z(t)$ and $h(t)$ be continuous and nonnegative functions de-*

fined for $t \geq 0$, $\varphi \in \mathbf{\Phi}$ and $c_i \in \mathbb{R}$, $i = 1, 2, 3$. Then

$$z(t) \leq c_1 + c_2 t^\gamma + c_3 t^\gamma \int_0^t h(s) z(s)\, ds, \quad \gamma,\ t \geq 0,$$

implies

$$z(t) \leq \begin{cases} (|c_1| + |c_2|) \exp\left(|c_3| \int_0^t h(s) ds\right), & 0 \leq t < 1 \\ A t^\gamma \exp\left(|c_3| \int_1^t s^\gamma h(s) ds\right), & t \geq 1, \end{cases}$$

where

$$A = |c_1| + |c_2| + |c_3|\, K \int_0^1 h(s) ds,$$

and

$$K = (|c_1| + |c_2|) \exp\left(|c_3| \int_0^1 h(s) ds\right) < \infty.$$

Lemma 3.2.3 *Let $z(t)$ satisfy*

$$z(t) \leq c_1 t^\gamma + c_2 t^\gamma \int_0^t \left[F_1(s, c_3 + z(s)) + F_2(s, c_4 + z(s)) + h(s)\right] ds, \quad t \geq 0, \tag{3.21}$$

where γ, $c_i > 0$, $i = 1, 2, 3, 4$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $F_j \in \mathbf{\Psi}$, $j = 1, 2$. Then

$$z(t) \leq t^\gamma g(t), \quad t > 0, \tag{3.22}$$

where

$$g(t) = \left(c_1 + c_2 \int_0^t \left[F_1(s, c_3) + F_2(s, c_4) + h(s)\right] ds\right) \times \exp\left(c_2 \int_0^t s^\gamma \left[N_1(s) + N_2(s)\right] ds\right), \quad t > 0, \tag{3.23}$$

with N_1 and N_2 are the functions in the definition of $\mathbf{\Psi}$ corresponding to F_1 and F_2, respectively.

Proof. Adding and subtracting $F_1(s, c_3)$ and $F_2(s, c_4)$ in the integrand of the relation (3.21) gives

$$\frac{z(t)}{t^\gamma} \leq c_1 + c_2 \int_0^t \left[F_1(s, c_3 + z(s)) + F_2(s, c_4 + z(s)) + h(s)\right] ds$$

$$= c_1 + c_2 \int_0^t [F_1(s, c_3 + z(s)) - F_1(s, c_3) + F_1(s, c_3)$$

$$+ F_2(s, c_4 + z(s)) - F_2(s, c_4) + F_2(s, c_4) + h(s)] \, ds$$

$$= c_1 + c_2 \int_0^t \left[F_1(s, c_3) + F_2(s, c_4) + h(s)\right] ds$$

$$+ c_2 \int_0^t \left[F_1(s, c_3 + z(s)) - F_1(s, c_3) + F_2(s, c_4 + z(s)) - F_2(s, c_4)\right] ds, \ \ t > 0. \tag{3.24}$$

As the functions F_i, $i = 1, 2$ are in $\mathbf{\Psi}$ we can write

$$\frac{z(t)}{t^\gamma} \leq c_1 + c_2 \int_0^t \left[F_1(s, c_3) + F_2(s, c_4) + h(s)\right] ds + c_2 \int_0^t \left[N_1(s) + N_2(s)\right] z(s) \, ds$$

$$= g_1(t) + c_2 \int_0^t s^\gamma \left[N_1(s) + N_2(s)\right] \frac{z(s)}{s^\gamma} ds, \quad t > 0, \tag{3.25}$$

where

$$g_1(t) = c_1 + c_2 \int_0^t \left[F_1(s, c_3) + F_2(s, c_4) + h(s)\right] ds, \quad t > 0. \tag{3.26}$$

Clearly g_1 is a continuous, positive and nondecreasing function defined for all

$t > 0$. Applying Theorem 2.4.3 to (3.25), we get

$$\frac{z(t)}{t^{\gamma}} \leq g_1(t) \exp\left(c_2 \int_0^t s^{\gamma} [N_1(s) + N_2(s)]\, ds\right), \quad t > 0, \tag{3.27}$$

and (3.22) follows directly from (3.26) and (3.27).

∎

3.3 Fractional differential problems with Riemann-Liouville fractional derivative

In this section, we consider the following fractional differential equation

$$D_0^{1+\alpha} y(t) = f(t, y(t), D_0^{\beta} y(t)), \quad t > 0, \tag{3.28}$$

with initial conditions

$$D_0^{\alpha} y(t)\,|_{t=0} = b_2 \text{ and } I_0^{1-\alpha} y(t)\,|_{t=0} = b_1, \quad b_1,\ b_2 \in \mathbb{R}, \tag{3.29}$$

where D_0^{σ} is the Riemann-Liouville fractional derivative of order $\sigma > 0$ and $0 \leq \beta < \alpha \leq 1$.

3.3.1 Problems with a non-fractional source

In this subsection, we study the asymptotic behavior of solutions of (3.28) when $\beta = 0$ and $0 < \alpha < 1$:

$$D_0^{1+\alpha} y\,(t) = f(t, y\,(t)), \quad 0 < \alpha \leq 1,\ t > 0, \tag{3.30}$$

with initial conditions

$$D_0^{\alpha} y\,(t)\,|_{t=0} = b_2 \text{ and } I_0^{1-\alpha} y\,(t)\,|_{t=0} = b_1, \quad b_1,\ b_2 \in \mathbb{R}, \tag{3.31}$$

in the space $C_{1-\alpha}^{1+\alpha}[0, \infty)$ defined by

$$C_{1-\alpha}^{1+\alpha}[0, \infty) = \left\{ y \in C_{1-\alpha}[0, \infty) : D_0^{1+\alpha} y \in C_{1-\alpha}[0, \infty) \right\}. \tag{3.32}$$

In the sequel, we suppose that the function $f(t, y)$ satisfies the following conditions

(A) $f(t, y) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is such that $f(., y\,(.)) \in C_{1-\alpha}[0, \infty)$ for any $y \in C_{1-\alpha}[0, \infty)$.

(B) There exist continuous functions $h,\ \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y\,(t))| \leq h(t)\varphi\left(t^{1-\alpha}\,|y(t)|\right),\ t \geq 0, \tag{3.33}$$

where $\varphi \in \mathbf{\Phi}$ and $h \in \mathbf{H}_1$.

The next result provides useful estimates for solutions of Problem (3.30)-(3.31).

Lemma 3.3.1 *Assume that* $y \in C_{1-\alpha}[0,\infty)$ *is a solution of (3.30)-(3.31) and* f *satisfies (**A**) and (**B**). Then, we have*

$$t^{1-\alpha}\left|y\left(t\right)\right| \leq z\left(t\right),\ t>0, \tag{3.34}$$

where

$$z\left(t\right)=\frac{\left|b_1\right|}{\Gamma(\alpha)}+\frac{\left|b_2\right|t}{\Gamma(\alpha+1)}+\frac{t}{\Gamma\left(\alpha+1\right)}\int_0^t h(s)\varphi\left(s^{1-\alpha}\left|y(s)\right|\right)ds,\ \ t>0. \tag{3.35}$$

Proof. Applying $I_0^{1+\alpha}$ to (3.30) we find

$$I_0^{1+\alpha}D_0^{1+\alpha}y\left(t\right)=I_0^{1+\alpha}f\left(t,y\left(t\right)\right),\ t>0.$$

Since $f \in C_{1-\alpha}[0,\infty)$, (3.30) implies that $D_0^{1+\alpha}y = D^2 I_0^{1-\alpha}y \in C_{1-\alpha}[0,\infty)$, then by Lemma 3.1.1, we have $I_0^{1-\alpha}y \in C_{1-\alpha}^2[0,\infty)$. As the hypotheses of Lemma 2.2.5 are fulfilled, we infer that

$$y\left(t\right)=\frac{b_1}{\Gamma(\alpha)}t^{\alpha-1}+\frac{b_2}{\Gamma(\alpha+1)}t^{\alpha}+I_0^{1+\alpha}f\left(t,y\left(t\right)\right),\quad t>0, \tag{3.36}$$

where b_1 and b_2 come from the initial conditions in (3.31). In view of (3.36) we deduce

$$\left|y\left(t\right)\right| \leq \frac{\left|b_1\right|}{\Gamma(\alpha)}t^{\alpha-1}+\frac{\left|b_2\right|}{\Gamma(\alpha+1)}t^{\alpha}+\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)}\int_0^t \left|f\left(s,y\left(s\right)\right)\right|ds,\quad t>0. \tag{3.37}$$

Multiplying both sides of (3.37) by $t^{1-\alpha}$ and using (3.33), we obtain the result. ∎

Theorem 3.3.1 *Let $y \in C_{1-\alpha}[0,\infty)$ be a solution of problem (3.30)-(3.31) and f satisfies (**A**) and (**B**). Then*

$$\lim_{t\to\infty} \frac{y(t)}{t^{\alpha}} = a \in \mathbb{R}.$$

Proof. It follows from Lemmas 3.2.2 and 3.3.1 that

$$\frac{|y(t)|}{t^{\alpha}} \leq H_0 =: G^{-1}(H), \quad t \geq 1, \tag{3.38}$$

where

$$H = G(A) + \frac{1}{\Gamma(1+\alpha)} \int_1^{\infty} sh(s)\, ds,$$

$$A = \frac{|b_1|}{\Gamma(\alpha)} + \frac{|b_2|}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \varphi\left(G^{-1}(K)\right) \int_0^1 h(s) ds,$$

and

$$K = G\left(\frac{|b_1|}{\Gamma(\alpha)} + \frac{|b_2|}{\Gamma(1+\alpha)}\right) + \frac{1}{\Gamma(1+\alpha)} \int_0^1 h(s) ds.$$

By (3.33), we see that

$$\left|\int_0^t f(s, y(s))\, ds\right| \leq \int_0^t |f(s, y(s))|\, ds \leq \int_0^t h(s)\varphi\left(s^{1-\alpha}|y(s)|\right) ds$$

$$\leq \int_0^1 h(s)\varphi\left(s^{1-\alpha}|y(s)|\right) ds + \int_1^t h(s)\varphi\left(s^{1-\alpha}|y(s)|\right) ds$$

$$\leq \int_0^1 h(s)\varphi(z(s))\, ds + \int_1^t sh(s)\varphi\left(\frac{|y(s)|}{s^{\alpha}}\right) ds, \ t > 0. \tag{3.39}$$

Therefore, from the first inequality of (3.17), (3.38) and (3.39) we obtain that

$$\left|\int_0^t f(s, y(s))\, ds\right| \leq \int_0^1 h(s)\varphi\left(G^{-1}(K)\right) ds + \int_1^t sh(s)\varphi(H_0)\, ds$$

$$\leq \varphi\left(G^{-1}(K)\right) \int_0^1 h(s) ds + \varphi(H_0) \int_1^t sh(s) ds < \infty,$$

because $h \in \mathbf{H}_1$, see (3.14). Thus, the integral $\int_0^t f(s, y(s))\, ds$ is absolutely convergent and consequently

$$\lim_{t\to\infty} \int_0^t f(s, y(s))\, ds < \infty. \tag{3.40}$$

Integrating both sides of (3.30), we find

$$D_0^\alpha y(t) = b_2 + \int_0^t f(s, y(s))\, ds, \quad t > 0.$$

In virtue of (3.40) we deduce that there exists $c \in \mathbb{R}$ such that

$$\lim_{t\to\infty} D_0^\alpha y(t) = c.$$

Further, by Lemma 3.1.5, we can write

$$\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \lim_{t\to\infty} \frac{D_0^\alpha y(t)}{\Gamma(\alpha+1)} = a,$$

and the proof is now complete. ∎

Example 3.3.1 *Consider the equation*

$$D_0^{1+\alpha} y(t) = t^{\gamma} e^{-t} (y(t))^{r}, \quad t > 0, \tag{3.41}$$

where $0 < \alpha < 1$*,* $0 < r < 1$*, and* $\gamma + 1 > (1-\alpha) r$*. Then, all solutions* $y \in C_{1-\alpha}[0,\infty)$ *of (3.41) enjoy the property* $\lim_{t\to\infty} \frac{y(t)}{t^{\alpha}} = a$ *for some real number* a*.*

Proof. We can rewrite (3.41) as follows

$$D_0^{1+\alpha} y(t) = t^{\gamma+(\alpha-1)r} e^{-t} \left(t^{1-\alpha} y(t)\right)^{r}.$$

Let $h(t) = t^{\gamma+(\alpha-1)r} e^{-t}$ and $\varphi(t) = t^{r}$. Then

$$\int_0^1 h(s)\, ds = \int_0^1 s^{\gamma+(\alpha-1)r} e^{-s} ds \leq \int_0^1 s^{\gamma+(\alpha-1)r} ds = \frac{1}{\gamma + (\alpha-1) r + 1} < \infty,$$

and

$$\int_1^{\infty} sh(s)\, ds < \int_0^{\infty} sh(s)\, ds = \int_0^{\infty} s^{\gamma+(\alpha-1)r+1} e^{-s} ds = \Gamma(\gamma + (\alpha-1) r + 2) < \infty.$$

Hence $h \in \mathbf{H}_1$, and φ is positive, continuous and nondecreasing function such that

$$\frac{1}{u}\varphi(v) = \frac{1}{u} v^{r} \leq \left(\frac{v}{u}\right)^{r} = \varphi\left(\frac{v}{u}\right), \quad v > 0,\ u > 1,$$

and

$$\int_{r_0}^{\infty} \frac{ds}{\varphi(s)} = \int_{r_0}^{\infty} \frac{ds}{s^r} = \infty, \ r_0 > 0.$$

Consequently $\varphi \in \mathbf{\Phi}$. Clearly, all conditions of Theorem 3.3.1 are satisfied and the result follows. ∎

Remark 3.3.1 *When $b_1 = 0$ in Theorem 3.3.1, then the condition ($\boldsymbol{B}$) is replaced by*

($\mathbf{B}'$) There exist continuous functions h, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y(t))| \leq h(t)\varphi\left(\frac{|y(t)|}{t^{\alpha}}\right), \quad t > 0,$$

where φ is positive and nondecreasing and h is such that

$$\int_1^{\infty} h(s)\, ds < \infty.$$

3.3.2 Equations with a fractional source term

Now we consider (3.28), with $0 < \beta < \alpha < 1$

$$D_0^{1+\alpha} y(t) = f(t, y(t), D_0^{\beta} y(t)), \quad t > 0, \tag{3.42}$$

with initial conditions

$$D_0^{\alpha} y(t)|_{t=0} = b_2 \text{ and } I_0^{1-\alpha} y(t)|_{t=0} = b_1, \quad b_1, \ b_2 \in \mathbb{R}, \tag{3.43}$$

in the space $C^{1+\alpha}_{1-\alpha}[0,\infty)$ defined in (3.32).

In the sequel, we suppose that the following conditions hold:

(A1) The function $f(t,u_1,u_2):(0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ is such that $f(.,u_1(.),u_2(.))\in C_{1-\alpha}[0,\infty)$ for any $u_1,u_2\in C_{1-\alpha}[0,\infty)$.

(A2) There exist continuous functions $h:\mathbb{R}_+\to\mathbb{R}_+$, $F_i:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+, i=1,2,$ such that

$$|f(t,u_1,u_2)|\leq F_1\left(t,t^{1-\alpha}|u_1|\right)+F_2\left(t,t^{1-(\alpha-\beta)}|u_2|\right)+h(t)\,,\ \ t>0, \quad (3.44)$$

where $F_i\in\mathbf{\Psi}$, $i=1,2$.

The next result provides useful estimates for solutions of problem (3.42)-(3.43).

Lemma 3.3.2 *Assume that $y\in C^{1+\alpha}_{1-\alpha}[0,\infty)$ is a solution of (3.42)-(3.43). Then, for all $t>0$, we have*

$$t^{1-(\alpha-\beta)}\left|D_0^\beta y(t)\right|\leq\frac{|b_1|}{\Gamma(\alpha-\beta)}+\frac{t}{\Gamma(1+\alpha-\beta)}\left(|b_2|+I_0^1\left|f(t,y(t),D_0^\beta y(t))\right|\right). \quad (3.45)$$

Proof. Note that $D_0^{1+\alpha}y=D^2I_0^{1-\alpha}y\in C_{1-\alpha}[0,\infty)$ implies $I_0^{1-\alpha}y\in C^2_{1-\alpha}[0,\infty)$, see Lemma 3.1.1. By Lemma 3.1.2, we see that

$$D_0^\beta y(t)=\frac{I_0^{1-\alpha}y(0)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}+I_0^{\alpha-\beta}D_0^\alpha y(t)\,,\quad t>0. \quad (3.46)$$

Integrating both sides of (3.42), we find

$$D_0^{\alpha} y(t) = b_2 + I_0^1 f(t, y(t), D_0^{\beta} y(t)), \quad t > 0. \tag{3.47}$$

Let us insert the expression (3.47) into (3.46), use Property 2.2.1, we obtain

$$D_0^{\beta} y(t) = \frac{b_1}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + I_0^{\alpha-\beta}\left(b_2 + I_0^1 f\left(s, y(s), D_0^{\beta} y(s)\right)\right)(t)$$

$$= \frac{b_1}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + \frac{b_2}{\Gamma(1+\alpha-\beta)} t^{\alpha-\beta} + I_0^{\alpha-\beta} I_0^1 f\left(t, y(t), D_0^{\beta} y(t)\right), \quad t > 0. \tag{3.48}$$

We deduce the bound

$$\left|D_0^{\beta} y(t)\right| \leq \frac{|b_1|\, t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{|b_2|\, t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + \frac{t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} I_0^1 \left|f\left(t, y(t), D_0^{\beta} y(t)\right)\right|, \ t > 0. \tag{3.49}$$

Multiplying both sides of (3.49) by $t^{1-(\alpha-\beta)}$ and the result follows. ∎

Lemma 3.3.3 *Assume that $y \in C_{1-\alpha}[0,\infty)$ is a solution of (3.42)-(3.43), f satisfies **(A1)**, **(A2)** with*

$$\int_0^{\infty} F_i\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) ds < \infty, \quad \int_0^{\infty} s N_i(s)\, ds < \infty, \ i = 1, 2, \tag{3.50}$$

and $h \in L_1(0,\infty)$ where N_i, $i = 1, 2$, are as in the definition of $\mathbf{\Psi}$. Then

$$\lim_{t\to\infty} \int_0^t f\left(s, y(s), D_0^{\beta} y(s)\right) ds < \infty.$$

Proof. Let

$$z(t) = \frac{t}{\Gamma(1+\alpha-\beta)}$$

$$\times \left(|b_2| + \int_0^t \left[F_1\left(s, s^{1-\alpha}|y(s)|\right) + F_2\left(s, s^{1-(\alpha-\beta)}\left|D_0^{\beta} y(s)\right|\right) + h(s) \right] \right) ds,\ t>0 \tag{3.51}$$

Then from Lemma 3.3.2 we get

$$t^{1-(\alpha-\beta)}\left|D_0^{\beta} y(t)\right| \le \frac{|b_1|}{\Gamma(\alpha-\beta)} + z(t),\ t>0. \tag{3.52}$$

Thus

$$F_1\left(t, t^{1-\alpha}|y(t)|\right) \le F_1\left(t, \frac{|b_1|}{\Gamma(\alpha)} + z(t)\right),\ t>0, \tag{3.53}$$

and

$$F_2\left(t, t^{1-(\alpha-\beta)}\left|D_0^{\beta} y(t)\right|\right) \le F_2\left(t, \frac{|b_1|}{\Gamma(\alpha-\beta)} + z(t)\right) \le F_2\left(t, \frac{|b_1|}{\Gamma(\alpha)} + z(t)\right),\ t>0. \tag{3.54}$$

Taking into account (3.51), (3.53) and (3.54) we are lead to

$$z(t) \le \frac{t}{\Gamma(1+\alpha-\beta)}$$

$$\times \left(|b_2| + \int_0^t \left[F_1\left(s, \frac{|b_1|}{\Gamma(\alpha)} + z(s)\right) + F_2\left(s, \frac{|b_1|}{\Gamma(\alpha)} + z(s)\right) + h(s) \right] ds \right),\ t>0. \tag{3.55}$$

Therefore, by Lemma 3.2.3, we have

$$z(t) \le Ct\ , \quad t>0, \tag{3.56}$$

where

$$C=\frac{1}{\Gamma\left(1+\alpha-\beta\right)}\left(\left|b_2\right|+\int_0^{\infty}\left[F_1\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}\right)+F_2\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}\right)+h\left(s\right)\right]ds\right)$$

$$\times\exp\left(\frac{1}{\Gamma\left(1+\alpha-\beta\right)}\int_0^{\infty}s\left[N_1\left(s\right)+N_2\left(s\right)\right]ds\right)<\infty.$$

It follows from (3.52) and (3.56) that

$$t^{1-(\alpha-\beta)}\left|D_0^{\beta}y\left(t\right)\right|\leq\frac{\left|b_1\right|}{\Gamma(\alpha-\beta)}+Ct\ ,\quad t>0. \tag{3.57}$$

On the other hand, again by our assumption (3.44) we see that

$$\left|\int_0^t f\left(s,y\left(s\right),D_0^{\beta}y\left(s\right)\right)ds\right|\leq\int_0^t\left|f\left(s,y\left(s\right),D_0^{\beta}y\left(s\right)\right)\right|ds$$

$$\leq\int_0^t\left[F_1\left(s,s^{1-\alpha}\left|y\left(s\right)\right|\right)+F_2\left(s^{1-(\alpha-\beta)}\left|D_0^{\beta}y\left(s\right)\right|\right)+h\left(s\right)\right]ds,\quad t>0. \tag{3.58}$$

Therefore from (3.57) and (3.58) we deduce that

$$\left|\int_0^t f\left(s,y\left(s\right),D_0^{\beta}y\left(s\right)\right)ds\right|\leq\int_0^t\left[F_1\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}+Cs\ \right)\right.$$

$$\left.+F_2\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}+Cs\ \right)+h\left(s\right)\right]ds$$

$$=\int_0^t\left[F_1\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}+Cs\ \right)-F_1\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}\right)+F_1\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}\right)\right.$$

$$\left.+F_2\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}+Cs\right)-F_2\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}\right)+F_2\left(s,\frac{\left|b_1\right|}{\Gamma(\alpha)}\right)\ \ +h\left(s\right)\right]ds,\ t>0 \tag{3.59}$$

As the functions F_i, $i = 1, 2$ are in $\mathbf{\Psi}$ we can write

$$\left| \int_0^t f\left(s, y(s), D_0^\beta y(s)\right) ds \right| \leq C \int_0^t s \left[N_1(s) + N_2(s)\right] ds$$

$$+ \int_0^t \left[F_1\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) + F_2\left(s, \frac{|b_1|}{\Gamma(\alpha)}\right) + h(s) \right] ds, \quad t > 0.$$

That is

$$\left| \int_0^t f\left(s, y(s), D_0^\beta y(s)\right) ds \right| < \infty, \quad t \geq 0.$$

The integral $\int_0^t f\left(s, y(s), D_0^\beta y(s)\right) ds$ is therefore absolutely convergent and the result follows. ∎

Theorem 3.3.2 *Under the same hypotheses as in Lemma 3.3.3 any solution* $y \in C_{1-\alpha}[0, \infty)$ *of problem (3.42)-(3.43) has the following property*

$$\lim_{t \to \infty} \frac{y(t)}{t^\alpha} = a, \ \ a \in \mathbb{R}.$$

Proof. It is clear, by virtue of (3.47) and Lemma 3.3.3, that there exists $b \in \mathbb{R}$ such that

$$\lim_{t \to \infty} D_0^\alpha y(t) = b.$$

Noting that Lemma 3.1.5 remains true for the new problem, we conclude that

$$\lim_{t \to \infty} \frac{y(t)}{t^\alpha} = \lim_{t \to \infty} \frac{D_0^\alpha y(t)}{\Gamma(1 + \alpha)} = a,$$

for some $a \in \mathbb{R}$. ∎

Remark 3.3.2 *If $b_1 = 0$ in Theorem 3.3.2, then we replace (3.44) and (3.50) by*

$$|f(t,u,v)| \leq F_1\left(t, \frac{|u|}{t^{\alpha}}\right) + F_2\left(t, \frac{|v|}{t^{\alpha-\beta}}\right),$$

and

$$\int_0^{\infty} N_i\,(s)\,ds < \infty,$$

respectively.

3.4 Fractional differential problems with Caputo fractional derivative

3.4.1 Problems with a non-fractional source

In this subsection, we study the asymptotic behavior of solutions of the following nonlinear fractional differential problem

$$\left({}^{C}D_0^{\alpha} y\right)'(t) = f(t, y\,(t)),\ \ 0 < \alpha < 1,\ t \geq 0, \tag{3.60}$$

with initial conditions

$$b_2 = \ {}^{C}D_0^{\alpha} y\,(t)\,|_{t=0} \ \text{ and } b_1 = y\,(0)\,, \tag{3.61}$$

in the space $C^{\alpha,1}_{1-\alpha}[0,\infty)$ defined by

$$C^{\alpha,1}_{1-\alpha}[0,\infty)=\left\{y\in AC[0,\infty),\left({}^{C}D^{\alpha}_{0}y\right)^{'}\in C_{1-\alpha}[0,\infty)\right\}. \tag{3.62}$$

In the sequel, we suppose that the function $f(t,y)$ satisfies the following conditions

(C1) The function $f(t,y):[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is such that $f(.,y\,(.))\in C_{1-\alpha}[0,\infty)$ for any $y\in AC[0,\infty)$.

(C2) There exist continuous functions $P,\varphi:\mathbb{R}_+\to\mathbb{R}_+$ ($\mathbb{R}_+=[0,\infty)$) such that

$$|f(t,y\,(t))|\leq P(t)\varphi\left(|y(t)|\right),\ t\geq 0, \tag{3.63}$$

where $\varphi\in\mathbf{\Phi}$ and P is such that

$$\int_1^{\infty}s^{\alpha}P\,(s)\,ds<\infty. \tag{3.64}$$

Theorem 3.4.1 *Let $y\in AC[0,\infty)$ be a solution of Problem (3.60)-(3.61) and f satisfies **(C1)**-**(C2)**. Then*

$$\lim_{t\to\infty}\frac{y\,(t)}{t^{\alpha}}=a\in\mathbb{R},\ \ \textit{as}\ t\to\infty.$$

Proof. From (3.10), with $D^{\alpha}_{0}y'\,(t)=DI^{1-\alpha}_{0}y'\,(t)=\left({}^{C}D^{\alpha}_{0}y\right)^{'}(t)=f(t,y\,(t))$, we deduce

$$y(t) = b_1 + \frac{b_2}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha} f(s, y(s))\, ds,\ t \geq 0. \tag{3.65}$$

By using (3.63) we have the bound

$$|y(t)| \leq |b_1| + \frac{|b_2|}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \int_0^t P(s) \varphi(|y(s)|)\, ds,\ t \geq 0. \tag{3.66}$$

Applying Lemma 3.2.2 to (3.66) we obtain

$$|y(t)| \leq \begin{cases} G^{-1}\left(G\left(|b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}\right) + \frac{1}{\Gamma(\alpha+1)} \int_0^t P(s)\, ds\right),\ 0 \leq t < 1 \\ t^{\alpha} G^{-1}\left(G(A) + \frac{1}{\Gamma(\alpha+1)} \int_1^t s^{\alpha} P(s)\, ds\right),\ t \geq 1, \end{cases}$$

where

$$A = |b_1| + \frac{|b_2|}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \varphi\left(G^{-1}(K)\right) \int_0^1 P(s) ds,$$

$$K = G\left(|b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}\right) + \frac{1}{\Gamma(\alpha+1)} \int_0^1 P(s) ds < \infty.$$

In virtue of (3.64) and P is continuous on $\mathbb{R}_+$, we set

$$|y(t)| \leq \begin{cases} C_1,\ 0 \leq t < 1, \\ t^{\alpha} C_2,\ t \geq 1, \end{cases} \tag{3.67}$$

with

$$C_1 = G^{-1}\left(G\left(|b_1| + \frac{|b_2|}{\Gamma(\alpha+1)}\right) + \frac{1}{\Gamma(\alpha+1)} \int_0^1 P(s)\, ds\right) < \infty,$$

and

$$C_2 = G^{-1}\left(G\left(A\right) + \frac{1}{\Gamma\left(\alpha+1\right)}\int_1^{\infty} s^{\alpha} P\left(s\right) ds\right) < \infty.$$

Next, we have

$$\int_0^t \left|f\left(s, y\left(s\right)\right)\right| ds \le \int_0^t P(s)\varphi\left(|y(s)|\right) ds$$

$$\le \int_0^1 P(s)\varphi\left(|y(s)|\right) ds + \int_1^t P(s)\varphi\left(|y(s)|\right) ds$$

$$\le \int_0^1 P(s)\varphi\left(|y(s)|\right) ds + \int_1^t s^{\alpha} P(s)\varphi\left(\frac{|y(s)|}{s^{\alpha}}\right) ds,\ t > 0. \tag{3.68}$$

In virtue of (3.67) and (3.68) we have that the integral $\int_0^t f\left(s, y\left(s\right)\right) ds$ is absolutely convergent and consequently

$$\lim_{t\to\infty} \int_0^t f\left(s, y\left(s\right)\right) ds < \infty. \tag{3.69}$$

Integrating (3.60), we obtain

$${}^{C}D_0^{\alpha} y\left(t\right) = b_2 + \int_0^t f\left(s, y\left(s\right)\right) ds,\ t > 0. \tag{3.70}$$

The relations (3.69) and (3.70) ensure the existence of $c \in \mathbb{R}$ such that

$$\lim_{t\to\infty} {}^{C}D_0^{\alpha} y\left(t\right) = c.$$

Further, by Lemma 3.1.6, we can write

$$\lim_{t\to\infty} \frac{y\left(t\right)}{t^{\alpha}} = \lim_{t\to\infty} \frac{{}^{C}D_0^{\alpha} y\left(t\right)}{\Gamma\left(\alpha+1\right)} = a,$$

and the proof is now complete.

∎

Example 3.4.1 *Consider the equation*

$$\left({}^{C}D_{0}^{\alpha}y\right)'(t) = e^{-t}\left(y(t)\right)^{r}, \ t > 0, \ 0 < \alpha, r \leq 1. \tag{3.71}$$

Then all solutions y of (3.71) have the property $\lim_{t\to\infty}\frac{y(t)}{t^{\alpha}} = a$, *as* $t \to \infty$.

Proof. Let $P(t) = e^{-t}$ and $\varphi(t) = t^{r}$. Then

$$\int_{1}^{\infty} s^{\alpha}P(s)\,ds \leq \int_{0}^{\infty} s^{\alpha}e^{-s}ds = \Gamma(\alpha+1) < \infty.$$

Note that φ is positive, continuous and nondecreasing function such that

$$u\varphi(v) = uv^{r} \leq (uv)^{r} = \varphi(uv), \ v > 0, \ u \geq 1$$

and

$$\int_{0}^{\infty}\frac{ds}{\varphi(s)} = \int_{0}^{\infty}\frac{ds}{s^{r}} = \infty,$$

therefore $\varphi \in \mathbf{\Phi}$. Thus we have proved that all conditions of Theorem 3.4.1 are satisfied. It results that every solution y of (3.71) enjoys the property $\lim_{t\to\infty}\frac{y(t)}{t^{\alpha}} = a$, as $t \to \infty$, $a \in \mathbb{R}$. ∎

Remark 3.4.1 *We point out here that problems (3.41) and (3.71) have global solutions for $0 < r < 1$. Whereas for $r > 1$ solutions may blow up in finite time.*

3.4.2 Equations with lower order fractional derivative

In this section, we study the asymptotic behavior of solutions of

$$\left({}^{C}D_{0}^{\alpha}y\right)'(t)=f(t,y\left(t\right),{}^{C}D_{0}^{\beta}y\left(t\right)),\ 0<\beta<\alpha<1,\, t>0, \tag{3.72}$$

with initial conditions

$$b_2 = {}^{C}D_{0}^{\alpha}y\left(t\right)|_{t=0}\ \text{ and } b_1=y\left(t\right)|_{t=0},\ \ b_1,\ b_2\in\mathbb{R}, \tag{3.73}$$

in the space $C_{1-\alpha}^{\alpha,1}[0,\infty)$ defined in (3.62).

We suppose that the function f satisfies the following conditions

(C3) The function $f(t,u,v):[0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ is such that $f(.,u\left(.\right),v\left(.\right))\in C_{1-\alpha}[0,\infty)$ for any $u,v\in AC[0,\infty)$.

(C4) There exist continuous functions $F_i:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$, such that

$$\left|f(t,u\left(t\right),v\left(t\right))\right|\le F_1\left(t,\left|u\left(t\right)\right|\right)+F_2\left(t,t^{\beta}\left|v\left(t\right)\right|\right),\ t\ge 0, \tag{3.74}$$

where $F_i\in\mathbf{\Psi}$, $i=1,2$.

The next result provides useful estimates for solutions of Problem (3.72)-(3.73).

Lemma 3.4.1 *Assume that $y\in AC[0,\infty)$ is solution of (3.72)-(3.73) and f satisfies **(C3)**-**(C4)**. Then, we have*

$$\max\left\{\left|y\left(t\right)\right|,t^{\beta}\left|{}^{C}D_{0}^{\beta}y\left(t\right)\right|\ \right\}\le\left|b_1\right|+z\left(t\right),\quad t>0, \tag{3.75}$$

where

$$z(t) = C_2 t^\alpha + C_3 t^\alpha \int_0^t \left[F_1(s, |y(s)|) + F_2\left(s, t^\beta \left| {}^C D_0^\beta y(s) \right| \right) \right] ds, \;\; t > 0, \quad (3.76)$$

and

$$C_3 = \max \left\{ \frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha-\beta+1)} \right\}, \;\; C_2 = |b_2| \, C_3.$$

Proof. Applying I_0^1 to (3.72) we obtain

$$\begin{aligned} {}^C D_0^\alpha y(t) &= b_2 + I_0^1 f\left(t, y(t), \; {}^C D_0^\beta y(t)\right) \\ &= b_2 + \int_0^t f\left(s, y(s), \; {}^C D_0^\beta y(s)\right) ds, \quad t > 0. \end{aligned} \quad (3.77)$$

Next, we apply I_0^α to both sides of (3.77), use Property 2.2.1 and Lemmas 2.2.2 and 2.3.1, we find

$$y(t) = b_1 + \frac{b_2}{\Gamma(\alpha+1)} t^\alpha + I_0^{1+\alpha} f\left(t, y(t), \; {}^C D_0^\beta y(t)\right)$$

$$= b_1 + \frac{b_2}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^\alpha f\left(s, y(s), \; {}^C D_0^\beta y(s)\right) ds, \;\; t > 0. \quad (3.78)$$

Thus from (3.78) and (3.74) we have the bound

$$|y(t)| \le |b_1| + \frac{|b_2|}{\Gamma(\alpha+1)} t^\alpha + \frac{t^\alpha}{\Gamma(\alpha+1)} \int_0^t \left| f\left(s, y(s), \; {}^C D_0^\beta y(s)\right) \right| ds$$

$$\le |b_1| + C_2 t^\alpha + C_3 t^\alpha \int_0^t \left(F_1(s, |y(s)|) + F_2\left(s, s^\beta \left| {}^C D_0^\beta y(s) \right| \right) \right) ds, \;\; t > 0. \quad (3.79)$$

By Lemma 3.1.3, we see that

$$ {}^{C}D_{0}^{\beta}y(t) = I_{0}^{\alpha-\beta}\left({}^{C}D_{0}^{\alpha}y(t)\right). $$

Let us insert the expression (3.77) into this last identity, use Property 2.2.1 and Lemma 2.2.2,

$$ {}^{C}D_{0}^{\beta}y(t) = I_{0}^{\alpha-\beta}\left(b_2 + I_{0}^{1} f\left(s, y(s),\ {}^{C}D_{0}^{\beta}y(s)\right)\right)(t) $$

$$ = \frac{b_2}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta} + I_{0}^{\alpha-\beta+1} f\left(t, y(t),\ {}^{C}D_{0}^{\beta}y(t)\right) $$

$$ = \frac{b_2}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta} + \frac{1}{\Gamma(\alpha-\beta+1)} \int_{0}^{t} (t-s)^{\alpha-\beta} f\left(s, y(s),\ {}^{C}D_{0+}^{\beta}y(s)\right) ds,\ t > 0. \tag{3.80} $$

Thus from (3.80) and (3.74) we obtain the bound

$$ t^{\beta}\left|{}^{C}D_{0}^{\beta}y(t)\right| \leq C_3 |b_2| t^{\alpha} + C_3 t^{\alpha} \int_{0}^{t} \left|f\left(s, y(s),\ {}^{C}D_{0+}^{\beta}y(s)\right)\right| ds $$

$$ \leq C_2 t^{\alpha} + C_3 t^{\alpha} \int_{0}^{t} \left(F_1(s, |y(s)|) + F_2\left(s, s^{\beta}\left|{}^{C}D_{0}^{\beta}y(s)\right|\right)\right) ds, \quad t > 0. \tag{3.81} $$

Therefore (3.75) follows directly from (3.76), (3.79) and (3.81).

∎

Theorem 3.4.2 *Suppose that f satisfies **(C3)**-**(C4)** and*

$$ \int_{0}^{\infty} s^{\alpha} N_i(s)\, ds < \infty,\ \int_{0}^{\infty} F_i(s, |b_1|)\, ds < \infty,\ i = 1, 2. \tag{3.82} $$

Then any solution $y(t) \in AC[0,\infty)$ of equation (3.72) has the following property

$$\lim_{t\to\infty} \frac{y(t)}{t^{\alpha}} = a \text{ for some } a \in \mathbb{R}, \text{ as } t \to \infty.$$

Proof. By virtue of Lemma 3.4.1 we get

$$F_1(t, |y(t)|) \le F_1(t, |b_1| + z(t)), \; t > 0, \tag{3.83}$$

and

$$F_2\left(t, t^{\beta} \left|{}^{C}D_0^{\beta} y(t)\right|\right) \le F_2(t, |b_1| + z(t)), \; t > 0. \tag{3.84}$$

Taking into account (3.76), (3.83) and (3.84) we are lead to

$$z(t) \le C_2 t^{\alpha} + C_3 t^{\alpha} \int_0^t [F_1(s, |b_1| + z(s)) + F_2(s, |b_1| + z(s))] \, ds, \; t > 0.$$

Therefore, by Lemma 3.2.3, we have

$$z(t) \le C t^{\alpha}, \; t > 0 \tag{3.85}$$

where

$$\begin{aligned} C = {} & \left(C_2 + C_3 \int_0^{\infty} [F_1(s, |b_1|) + F_2(s, |b_1|)] \, ds\right) \\ & \times \exp\left(C_3 \int_0^{\infty} s^{\gamma} [N_1(s) + N_2(s)] \, ds\right) < \infty. \end{aligned}$$

It follows from Lemma 3.4.1 and (3.85) that

$$|y(t)| \leq |b_1| + Ct^{\alpha} \quad \text{and } t^{\beta} \left| {}^{C}D_{0}^{\beta} y(t) \right| \leq |b_1| + Ct^{\alpha}, \quad t > 0. \tag{3.86}$$

On the other hand, again by our assumption (3.74) we see that

$$\left| \int_0^t f\left(s, y(s), \, {}^{C}D_{0}^{\beta} y(s)\right) ds \right| \leq \int_0^t \left| f\left(s, y(s), \, {}^{C}D_{0}^{\beta} y(s)\right) \right| ds$$

$$\leq \int_0^t \left[F_1(s, |y(s)|) + F_2\left(s, s^{\beta} \left| {}^{C}D_{0}^{\beta} y(s) \right| \right) \right] ds, \ t > 0. \tag{3.87}$$

Therefore from (3.86) and (3.87) we obtain

$$\left| \int_0^t f\left(s, y(s), \, {}^{C}D_{0}^{\beta} y(s)\right) ds \right| \leq \int_0^t \left[F_1(s, |b_1| + Cs^{\alpha}) + F_2(s, |b_1| + Cs^{\alpha}) \right] ds$$

$$= \int_0^t \{ F_1(s, |b_1| + Cs^{\alpha}) - F_1(s, |b_1|) + F_1(s, |b_1|)$$

$$+ F_2(s, |b_1| + Cs^{\alpha}) - F_2(s, |b_1|) + F_2(s, |b_1|) \} \, ds, \ t > 0.$$

As the functions F_i, $i = 1, 2$ are in $\mathbf{\Psi}$ we can write

$$\left| \int_0^t f\left(s, y(s), \, {}^{C}D_{0}^{\beta} y(s)\right) ds \right|$$

$$\leq C \int_0^t s^{\alpha} \left[N_1(s) + N_s(s) \right] ds + \int_0^t \left[F_1(s, |b_1|) + F_2(s, |b_1|) \right] ds < \infty, \tag{3.88}$$

where we have used (3.82). Thus the integral $\int_0^t f\left(s, y(s), \, {}^{C}D_{0}^{\beta} y(s)\right) ds$ is absolutely convergent and consequently

$$\lim_{t\to\infty}\int_0^t f\left(s, y\left(s\right),\ {}^{C}D_0^{\beta}y\left(s\right)\right)ds < \infty.$$

By (3.77) we have that there exists $b \in \mathbb{R}$ such that

$$\lim_{t\to\infty}\ {}^{C}D_0^{\alpha}y\left(t\right) = b.$$

Further, by Lemma 3.1.6, we conclude that

$$\lim_{t\to\infty}\frac{y\left(t\right)}{t^{\alpha}} = \lim_{t\to\infty}\frac{{}^{C}D_0^{\alpha}y\left(t\right)}{\Gamma\left(\alpha+1\right)} = a,$$

and the proof is now complete. ∎

CHAPTER 4

POWER TYPE DECAY AND BOUNDEDNESS FOR A GENERAL FRACTIONAL DIFFERENTIAL PROBLEM OF ORDER BETWEEN ZERO AND ONE

In this chapter, we consider the following fractional differential equation

$$D_0^{\alpha} y(t) = f\left(t, y(t), D_0^{\beta} y(t)\right), \ 0 < \beta < \alpha < 1, \ t > 0, \tag{4.1}$$

where f is a continuous nonlinear function and D_0^{σ} is either the Riemann-Liouville derivative or the Caputo derivative

We shall investigate the behavior of solutions of t ordinary fractional differential equation (4.1) with certain nonlinearities. We will determine sufficient conditions on the nonlinearities which allow us that solutions of (4.1) can be extended and decay for all time in a weighted space of continuous functions. Therefore, as time increases, solutions "decay" as a power-type function. We mention here that the right hand side of (4.1) contains "fractional derivatives". These "fractional derivatives" involve naturally (by definition) kernels of singular type.

4.1 Some important inequalities

In this section, we prove some lemmas and properties which will be used in our results later.

Remark 4.1.1 *We will use the following equivalency. If* p, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then for* $\alpha > 0$, $p(\alpha - 1) + 1 > 0 \Longleftrightarrow q\alpha > 1$.

Now we prove some lemmas which will be used to prove the main results.

Lemma 4.1.1 *: For all $\alpha > 0$ and $\beta > -1$, we have*

$$\int_0^t (t-s)^{\alpha-1} s^{\beta} ds = K_{\alpha,\beta} t^{\alpha+\beta}, \quad t \geq 0,$$

where

$$K_{\alpha,\beta} = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Lemma 4.1.2 *If υ, $\lambda + 1 > 1/q$, for some $q > 1$, and h is a nonnegative continuous function defined on $\mathbb{R}_+$, then*

$$\int_0^t (t-s)^{\upsilon-1} s^{\lambda} h(s)\, ds \leq C t^{\upsilon+\lambda-1/q} \left(\int_0^t h^q(s)\, ds \right)^{1/q}, \ t > 0, \tag{4.2}$$

where

$$C = K_{p(\upsilon-1)+1, p\lambda} = \frac{\Gamma(p(\upsilon-1)+1)\Gamma(p\lambda+1)}{\Gamma(p(\upsilon-1)+p\lambda+2)}, \ 1/p + 1/q = 1.$$

Proof. Thanks to Hölder's inequality, it is clear that

$$\int_0^t (t-s)^{\upsilon-1} s^{\lambda} h(s)\, ds \leq \left(\int_0^t (t-s)^{p(\upsilon-1)} s^{p\lambda} ds \right)^{1/p} \left(\int_0^t h^q(s)\, ds \right)^{1/q}, \ t > 0.$$

By virtue of Lemma 4.1.1, we obtain the result. ∎

Lemma 4.1.3 *If $w > 0$, υ, $\lambda > 1/q$, for some $q > 1$, and h nonnegative contin-*

uous function defined on $\mathbb{R}_+$, *then*

$$\int_0^t (t-s)^{\upsilon-1} s^{\lambda-1} e^{-ws} h(s)\, ds \le C t^{\upsilon-1} \left(\int_0^t h^q(s)\, ds \right)^{1/q}, \ t > 0, \tag{4.3}$$

where

$$C = \left[\max\left\{1, 2^{1-\lambda_1}\right\} \Gamma(\lambda_2) \left(1 + \frac{(\lambda_2)(\lambda_2+1)}{\lambda_1}\right) (pw)^{-\lambda_2} \right]^{1/p},$$

$$1/p + 1/q = 1, \ \lambda_1 = p(\upsilon - 1) + 1 \ \textit{and} \ \lambda_2 = p(\lambda - 1) + 1.$$

Proof. Applying Hölder inequality to the left hand side of (4.3) we obtain for $t > 0$

$$\int_0^t (t-s)^{\upsilon-1} s^{\lambda-1} e^{-ws} h(s)\, ds \le \left(\int_0^t (t-s)^{p(\upsilon-1)} s^{p(\lambda-1)} e^{-pws} \right)^{1/p} \left(\int_0^t h^q(s)\, ds \right)^{1/q}. \tag{4.4}$$

It follows from the hypothesis υ, $\lambda > 1/q$ that

$$p(\upsilon - 1) + 1 = \lambda_1 > 0 \text{ and } \lambda_2 = p(\lambda - 1) + 1 > 0.$$

Applying Lemma 2.4.3 to (4.4) (with υ replaced by λ_1, λ replaced by λ_2 and w replaced by pw), we obtain the result. ∎

The following Lemmas follow from Bihari inequality.

Lemma 4.1.4 *Let* z *and* h *be nonnegative continuous functions defined on* $\mathbb{R}_+$. *Let* $\varphi_i(z)$, $i = 1, 2$, *be continuous nondecreasing functions defined on* $\mathbb{R}_+$ *and*

$\varphi_i(z) > 0$ on $(0, \infty)$. If

$$z(t) \leq K_1 + K_2 \left(\int_0^t h^q(s)\, \varphi_1^q(z(s))\, \varphi_2^q(z(s))\, ds \right)^{\frac{1}{q}}, \ q > 1, \ t > 0, \tag{4.5}$$

where K_i, $i = 1, 2$, are nonnegative constants, then for $0 < t \leq T$

$$z(t) \leq \left[G^{-1} \left(G\left(2^{q-1} K_1\right) + 2^{q-1} K_2 \int_0^t h^q(s)\, ds \right) \right]^{1/q},$$

where

$$G(x) = \int_{x_0}^{x} \frac{ds}{\varphi_1^q\left(s^{1/q}\right) \varphi_2^q\left(s^{1/q}\right)}, \ x > x_0 > 0,$$

and G^{-1} is the inverse function of G, and $T > 0$ is chosen such that

$$G\left(2^{q-1} K_1\right) + 2^{q-1} K_2 \int_0^t h^q(s)\, ds \in Dom\left(G^{-1}\right),$$

for all $t > 0$ lying in the interval $0 < t \leq T$.

Proof. Raising both sides of (4.5) to the power q and using Lemma 2.4.4, we have

$$z^q(t) \leq B_1 + B_2 \int_0^t h^q(s)\, \varphi_1^q(z(s))\, \varphi_2^q(z(s))\, ds, \ t > 0, \tag{4.6}$$

where

$$B_i = 2^{q-1} K_i, \ i = 1, 2.$$

Now, let $u(t) = z^q(t)$, then (4.6) can be written as

$$u(t) \leq B_1 + B_2 \int_0^t h^q(s)\, g(u(s))\, ds, \ \ t > 0, \tag{4.7}$$

where

$$g(r) = \varphi_1^q\left(r^{1/q}\right) \varphi_2^q\left(r^{1/q}\right). \tag{4.8}$$

Since φ_i, $i = 1, 2$, are continuous and nondecreasing functions, then g is continuous and nondecreasing function. Applying Bihari's inequality (Theorem 2.4.1) to (4.7), we obtain the result. ∎

Lemma 4.1.5 *Let q, z, h_i and φ_i, $i = 1, 2$, be as in Lemma 4.1.4. If*

$$z(t) \leq K_1 + K_2 \left[\left(\int_0^t h_1^q(s)\, \varphi_1^q(z(s))\, ds\right)^{\frac{1}{q}} + \left(\int_0^t h_2^q(s)\, \varphi_2^q(z(s))\, ds\right)^{\frac{1}{q}}\right], \ t > 0, \tag{4.9}$$

then, for $0 < t \leq T$

$$z(t) \leq \left[G^{-1}\left(G\left(2^{q-1}K_1^q\right) + 2^{2(q-1)}K_2^q \int_0^t \left[h_1^q(s) + h_2^q(s)\right] ds\right)\right]^{1/q}, \tag{4.10}$$

where

$$G(x) = \int_{x_0}^x \frac{ds}{g(s)} = \int_{x_0}^x \frac{ds}{\varphi_1^q(s^{1/q}) + \varphi_2^q(s^{1/q})}, \ \ x > \ x_0 > 0,$$

and G^{-1} is the inverse function of G, and $T > 0$ is chosen such that

$$G\left(2^{q-1}K_1^q\right) + 2^{2(q-1)}K_2^q \int_0^t \left[h_1^q(s) + h_2^q(s)\right] ds \in Dom\left(G^{-1}\right)$$

for all $t > 0$ lying in the interval $0 < t \leq T$.

Proof. Raising both sides of (4.9) to the power q, we obtain

$$z^q(t) \leq B_1 + B_2 \left[\int_0^t h_1^q(s)\, \varphi_1^q(z(s))\, ds + \int_0^t h_2^q(s)\, \varphi_2^q(z(s))\, ds \right], \ t > 0, \quad (4.11)$$

where

$$B_1 = 2^{q-1} K_1^q \text{ and } B_2 = 2^{2(q-1)} K_2^q.$$

Furthermore, clearly we have

$$h_1^q(s)\, \varphi_1^q(z(s)) + h_2^q(s)\, \varphi_2^q(z(s)) \leq \left[h_1^q(s) + h_2^q(s)\right]\left[\varphi_1^q(z(s)) + \varphi_2^q(z(s))\right]. \quad (4.12)$$

Now, let $u(t) = z^q(t)$, then by (4.11) and (4.12) we can write

$$u(t) \leq B_1 + B_2 \int_0^t \left[h_1^q(s) + h_2^q(s)\right] g(u(s))\, ds, \ t > 0, \quad (4.13)$$

where

$$g(r) = \varphi_1^q\left(r^{1/q}\right) + \varphi_2^q\left(r^{1/q}\right). \quad (4.14)$$

Because φ_i, $i = 1, 2$, are continuous and nondecreasing functions, then g is continuous, nondecreasing function. Then, applying Bihari's inequality to (4.13), we obtain

$$u(t) \leq G^{-1}\left(G(B_1) + B_2 \int_0^t \left[h_1^q(s) + h_2^q(s)\right] ds \right), \quad t > 0.$$

∎

4.2 Power type decay of Riemann-Liouville fractional differential equation

In this section, we consider the following fractional differential problem

$$\begin{cases} D_0^{\alpha} y(t) = f\left(t, y(t), D_0^{\beta} y(t)\right), \ 0 < \beta < \alpha < 1, \ t > 0, \\ I_0^{1-\alpha} y(t)\,|_{t=0} = b, \end{cases} \tag{4.15}$$

where D_0^{σ} is the Riemann-Liouville fractional derivative in the space $C_{1-\alpha}^{\alpha}[0,\infty)$ defined by

$$C_{1-\alpha}^{\alpha}[0,\infty) = \left\{ y \in C_{1-\alpha}[0,\infty) : D_0^{\alpha} y \in C_{1-\alpha}[0,\infty) \right\}. \tag{4.16}$$

In the sequel, we consider the following assumption on f :

(A1) The function $f : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is such that $f(., u(.), v(.)) \in C_{1-\alpha}[0,\infty)$ for any u, $v \in C_{1-\alpha}[0,\infty)$.

In addition, we consider the following different bounds:

(B1) There exist continuous functions h, φ_1, $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|f(t, u(t), v(t))| \leq t^{\gamma} e^{-\delta t} h(t) \varphi_1\left(t^{1-\alpha} |u(t)|\right) \varphi_2\left(t^{1-(\alpha-\beta)} |v(t)|\right), \ t > 0, \tag{4.17}$$

where $h \in L_q(0,\infty)$ for some $q > \frac{1}{\alpha-\beta}$, $\gamma > -\frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\delta > 0$, and φ_i, $i = 1, 2$, are nondecreasing functions.

(B2) There exist continuous functions h_i, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|f(t,u(t),v(t))| \le t^{\gamma_1} e^{-\delta_1 t} h_1(t)\,\varphi_1\left(t^{1-\alpha}|u(t)|\right)$$

$$+\, t^{\gamma_2} e^{-\delta_2 t} h_2(t)\,\varphi_2\left(t^{1-(\alpha-\beta)}|v(t)|\right), \; t>0, \tag{4.18}$$

where $h_i \in L_q(0,\infty)$ for some $q > \frac{1}{\alpha-\beta}$, $\gamma_i > -\frac{1}{p}$, $\frac{1}{p}+\frac{1}{q}=1$, $\delta_i > 0$, and φ_i, $i=1,2$, are nondecreasing functions.

Remark 4.2.1 *Note that the class (4.17) of nonlinearities is different from the one in (3.44) (in addition to the order of the FDE). Although (4.17) may be rewritten in the form of (3.44) and F_i, $i=1,2$ are also nondecreasing, here we do not assume φ_i, $i=1,2$ satisfying the RHS of the definition of $\mathbf{\Psi}$.*

The next results provide useful estimates for solutions of Problem (4.15).

Lemma 4.2.1 *Assume that $y \in C_{1-\alpha}[0,\infty)$ is a solution of (4.15) and f satisfies (**A1**)-(**B1**). Then*

$$\max\left\{t^{1-\alpha}|y(t)|, t^{1-(\alpha-\beta)}\left|D^{\beta}y(t)\right|\right\} \le z(t), \quad t>0.$$

Here

$$z(t) = K_1 + K_2\left(\int_0^t h^q(s)\,\varphi_1^q\left(s^{1-\alpha}|y(s)|\right)\varphi_2^q\left(s^{1-(\alpha-\beta)}\left|D_0^{\beta}y(s)\right|\right)ds\right)^{\frac{1}{q}}, \; t>0, \tag{4.19}$$

where

$$K_1 = |b| \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}\right\},\ K_2 = \max\{C_1, C_2\},$$

$$C_1 = \frac{1}{\Gamma(\alpha)}\left(\max\left\{1, 2^{p(1-\alpha)}\right\}\Gamma(p\gamma+1)\left(1+\frac{(p\gamma+1)(p\gamma+2)}{p(\alpha-1)+1}\right)(p\delta)^{-(p\gamma+1)}\right)^{\frac{1}{p}},$$

and

$$C_2 = \frac{1}{\Gamma(\alpha-\beta)}\left(\max\left\{1, 2^{p(1-(\alpha-\beta))}\right\}\Gamma(p\gamma+1)\left(1+\frac{(p\gamma+1)(p\gamma+2)}{p(\alpha-\beta-1)+1}\right)(p\delta)^{-(p\gamma+1)}\right)^{\frac{1}{p}}.$$

Proof. Since $f \in C_{1-\alpha}[0,\infty)$, (4.15) implies that $D_0^\alpha y = DI_0^{1-\alpha} y \in C_{1-\alpha}[0,\infty)$, by Lemma 3.1.1, we have $I_0^{1-\alpha} y \in C_{1-\alpha}^1[0,\infty)$. Then the hypotheses of Lemma 2.2.5 are fulfilled. Applying I_0^α to (4.15) and using Lemma 2.2.5, we get

$$y(t) = \frac{b}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D_0^\beta y(s)\right) ds,\ t>0. \tag{4.20}$$

Next, multiplying both sides of (4.20) by $t^{1-\alpha}$ and using the inequality (4.17), we obtain

$$t^{1-\alpha}|y(t)| \le \frac{|b|}{\Gamma(\alpha)}$$
$$+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s^\gamma e^{-\delta s} h(s)\,\varphi_1\left(s^{1-\alpha}|y(s)|\right)\varphi_2\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right) ds,\ t>0. \tag{4.21}$$

In view of Lemma 4.1.3, we find

$$t^{1-\alpha}|y(t)| \le \frac{|b|}{\Gamma(\alpha)} + C_1\left(\int_0^t h^q(s)\,\varphi_1^q\left(s^{1-\alpha}|y(s)|\right)\varphi_2^q\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right) ds\right)^{\frac{1}{q}},\ t>0. \tag{4.22}$$

By Lemma 3.1.2, we see that

$$D_0^{\beta} y(t) = \frac{b}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} D_0^{\alpha} y(s)\, ds$$

$$= \frac{b}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f\left(s, y(s), D_0^{\beta} y(s)\right) ds,\ t>0. \tag{4.23}$$

Multiplying both sides of (4.23) by $t^{1-(\alpha-\beta)}$ and using the inequality (4.17), we deduce

$$t^{1-(\alpha-\beta)} \left| D_0^{\beta} y(t) \right| \leq \frac{|b|}{\Gamma(\alpha-\beta)} + \frac{t^{1-(\alpha-\beta)}}{\Gamma(\alpha-\beta)}$$

$$\times \int_0^t (t-s)^{\alpha-\beta-1} s^{\gamma} e^{-\delta s} h(s) \varphi_1\left(s^{1-\alpha} |y(s)|\right) \varphi_2\left(s^{1-(\alpha-\beta)} \left| D_0^{\beta} y(s) \right|\right) ds,\ t>0.$$

Again by Lemma 4.1.3, we conclude

$$t^{1-(\alpha-\beta)} \left| D_0^{\beta} y(t) \right| \leq \frac{|b|}{\Gamma(\alpha-\beta)}$$

$$+C_2 \left(\int_0^t h^q(s) \varphi_1^q\left(s^{1-\alpha} |y(s)|\right) \varphi_2^q\left(s^{1-(\alpha-\beta)} \left| D_0^{\beta} y(s) \right|\right) ds \right)^{\frac{1}{q}},\ t>0. \tag{4.24}$$

Therefore, the result follows from (4.19), (4.22) and (4.24). ∎

Lemma 4.2.2 *Assume that $y \in C_{1-\alpha}[0,\infty)$ is a solution of (4.15) and f satisfies (**A1**)-(**B2**). Then, we have*

$$\max\left\{ t^{1-\alpha} |y(t)|, t^{1-(\alpha-\beta)} \left| D^{\beta} y(t) \right| \right\} \leq z(t), \quad t>0,$$

where

$$z(t) = K_1 + K_2 \left[\left(\int_0^t h_1^q(s)\,\varphi_1^q\left(s^{1-\alpha}|y(s)|\right) ds \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^t h_2^q(s)\,\varphi_2^q\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right) ds \right)^{\frac{1}{q}} \right], \; t > 0 \tag{4.25}$$

$$K_1 = \max\left\{ \frac{|b|}{\Gamma(\alpha)}, \frac{|b|}{\Gamma(\alpha-\beta)} \right\}, \; K_2 = \max\{C_3, C_3'\},$$

$$C_3 = \max\{C_1, C_2\}, \; C_3' = \max\{C_1', C_2'\},$$

$$C_i = \frac{1}{\Gamma(\alpha)} \left(\max\left\{1, 2^{p(1-\alpha)}\right\} \Gamma(p\gamma_i + 1) \left(1 + \frac{p\gamma_i + 1}{p(\alpha-1)+1}\right) (p\delta_i)^{-(p\gamma_i+1)} \right)^{\frac{1}{p}},$$

and

$$C_i' = \frac{\left(\left[\max\left\{1, 2^{p(1-(\alpha-\beta))}\right\} \Gamma(p\gamma_i+1)\right]^{\frac{1}{p}} \right)}{\Gamma(\alpha-\beta)} \left[\left(1 + \frac{p\gamma_i+1}{p(\alpha-\beta-1)+1}\right) (p\delta_i)^{-(p\gamma_i+1)} \right]^{\frac{1}{p}}.$$

Proof. Multiplying both sides of (4.20) by $t^{1-\alpha}$ and using the inequality (4.18), we find

$$t^{1-\alpha}|y(t)| \le \frac{|b|}{\Gamma(\alpha)} + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\gamma_1} e^{-\delta_1 s} h_1(s)\,\varphi_1\left(s^{1-\alpha}|y(s)|\right) ds$$

$$+ \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\gamma_2} e^{-\delta_2 s} h_2(s)\,\varphi_2\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right) ds, \; t > 0.$$

Since $q > \frac{1}{\alpha-\beta}$, $\gamma_i > -\frac{1}{p}$ and $\delta_i > 0$, then $p(\alpha-1)+1 > 0$, $p\gamma_i + 1 > 0$ and $p\delta_i > 0$, $i = 1, 2$, so we can apply Lemma 4.1.3 to get

$$t^{1-\alpha}|y(t)| \le \frac{|b|}{\Gamma(\alpha)} + C_3 \left[\left(\int_0^t h_1^q(s)\,\varphi_1^q\left(s^{1-\alpha}|y(s)|\right) ds \right)^{\frac{1}{q}} \right.$$

$$+\left(\int_0^t h_2^q(s)\,\varphi_2^q\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right)ds\right)^{\frac{1}{q}}\Bigg],\ t>0. \tag{4.26}$$

Multiplying both sides of (4.23) by $t^{1-(\alpha-\beta)}$ and using the inequality (4.18), we obtain

$$t^{1-(\alpha-\beta)}\left|D_0^\beta y(t)\right| \le K_1+\frac{t^{1-(\alpha-\beta)}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}s^{\gamma_1}e^{-\delta_1 s}h_1(s)\,\varphi_1\left(s^{1-\alpha}|y(s)|\right)ds$$

$$+\frac{t^{1-(\alpha-\beta)}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}s^{\gamma_2}e^{-\delta_2 s}h_2(s)\,\varphi_2\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right)ds,\ t>0.$$

By virtue of Lemma 4.1.3, we can write

$$t^{1-(\alpha-\beta)}\left|D_0^\beta y(t)\right| \le K_1+C_3'\left[\left(\int_0^t h_1^q(s)\,\varphi_1^q\left(s^{1-\alpha}|y(s)|\right)ds\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^t h_2^q(s)\,\varphi_2^q\left(s^{1-(\alpha-\beta)}\left|D_0^\beta y(s)\right|\right)ds\right)^{\frac{1}{q}}\right],\ t>0. \tag{4.27}$$

Therefore, the result follows from (4.25), (4.26) and (4.27). ∎

Theorem 4.2.1 *Suppose that f satisfies (**A1**) and (**B1**), then, for any solution $y\in C_{1-\alpha}[0,\infty)$ of Problem (4.15), there exists a positive constant C such that*

$$|y(t)|\le Ct^{\alpha-1}\ \textit{and}\ \left|D_0^\beta y(t)\right|<Ct^{\alpha-\beta-1},\ t>0,$$

provided that

$$\int_{x_0}^\infty \frac{ds}{\varphi_1^q\left(s^{1/q}\right)\varphi_2^q\left(s^{1/q}\right)}=\infty,\ x_0>0.$$

Proof. From Lemma 4.2.1 we deduce

$$\varphi_1\left(t^{1-\alpha}\left|y(t)\right|\right) \leq \varphi_1(z(t)) \text{ and } \varphi_2\left(t^{1-(\alpha-\beta)}\left|D^\beta y(t)\right|\right) \leq \varphi_2(z(t)), \ t > 0, \tag{4.28}$$

where $z(t)$ is as in (4.19). Using the inequalities in (4.28), it follows from (4.19) that

$$z(t) \leq K_1 + K_2\left(\int_0^t h^q(s)\,\varphi_1^q(z(s))\,\varphi_2^q(z(s))\,ds\right)^{\frac{1}{q}}, \ t > 0.$$

Therefore, by Lemma 4.1.4, we obtain

$$z(t) \leq \left[G^{-1}\left(G\left(2^{q-1}K_1\right) + 2^{q-1}K_2\int_0^t h^q(s)\,ds\right)\right]^{1/q}, \ t > 0.$$

As $h \in L_q(0,\infty)$, we conclude

$$z(t) \leq C = \left[G^{-1}\left(G\left(2^{q-1}K_1\right) + 2^{q-1}K_2\int_0^\infty h^q(s)\,ds\right)\right]^{1/q} < \infty.$$

Again by Lemma 4.2.1 we find

$$|y(t)| \leq Ct^{\alpha-1} \text{ and } \left|D_0^\beta y(t)\right| < Ct^{\alpha-\beta-1}, \ t > 0.$$

∎

Theorem 4.2.2 *Suppose that f satisfies (**A1**) and (**B2**), then, for any solution* *of Problem (4.15), there exists a positive constant C such that*

$$|y(t)| \leq Ct^{\alpha-1} \text{ and } \left|D_0^\beta y(t)\right| < Ct^{\alpha-\beta-1},\ t > 0,$$

provided that

$$\int_{x_0}^{\infty} \frac{ds}{\varphi_1^q\left(s^{\frac{1}{q}}\right) + \varphi_2^q\left(s^{\frac{1}{q}}\right)} = \infty,\ x_0 > 0.$$

Proof. By virtue of Lemma 4.2.2 we see that

$$\varphi_1\left(t^{1-\alpha}|y(t)|\right) \leq \varphi_1(z(t)) \text{ and } \varphi_2\left(t^{1-(\alpha-\beta)}\left|D^\beta y(t)\right|\right) \leq \varphi_2(z(t)),\ t > 0. \tag{4.29}$$

Taking into account (4.25) and (4.29) we are lead to

$$z(t) \leq K_1 + K_2\left[\left(\int_0^t h_1^q(s)\,\varphi_1^q(z(s))\,ds\right)^{\frac{1}{q}} + \left(\int_0^t h_2^q(s)\,\varphi_2^q(z(s))\,ds\right)^{\frac{1}{q}}\right],\ t > 0.$$

Therefore, by Lemma 4.1.5, we find

$$z(t) \leq \left[G^{-1}\left(G\left(2^{q-1}K_1^q\ \right) + 2^{2(q-1)}K_2^q\int_0^t \left[h_1^q(s) + h_2^q(s)\right]ds\right)\right]^{1/q},\ t > 0.$$

As $h_i \in L_q(0,\infty)$, $i = 1, 2$, we deduce

$$z(t) \leq C = \left[G^{-1}\left(G\left(2^{q-1}K_1\right) + 2^{q-1}K_2\int_0^\infty h^q(s)\,ds\right)\right]^{1/q} < \infty.$$

Thus

$$|y(t)| \leq Ct^{\alpha-1} \text{ and } \left|D_0^\beta y(t)\right| < Ct^{\alpha-\beta-1},\ t > 0.$$

Example 4.2.1 *Consider the problem*

$$\begin{cases} D_0^{1/2} y\left(t\right) = t^2 e^{-2t} \left(\cos y^2\right) \left(y\left(t\right)\right)^{1/5} \left(D_0^{1/3} y\left(t\right)\right)^{1/3}, \ t > 0, \\ I^{1-1/2} y\left(t\right)|_{t=0} = b. \end{cases} \tag{4.30}$$

We can rewrite (4.30) as follows

$$D_0^{1/2} y\left(t\right) = t^{2+\frac{(1/2-1)}{5}+\frac{(1/2-1/3-1)}{3}} e^{-t} e^{-t} \left(\cos y^2\right) \left(t^{1-1/2} y\left(t\right)\right)^{1/5} \left(t^{1-(1/2-1/3)} D_0^{1/3} y\left(t\right)\right)^{1/3}$$

$$= t^{73/45} e^{-t} e^{-t} \left(\cos y^2\right) \left(t^{1-1/2} y\left(t\right)\right)^{1/5} \left(t^{1-(1/2-1/3)} D_0^{1/3} y\left(t\right)\right)^{1/3}, \ q > 4, \ t > 0.$$

Let $h\left(t\right) = e^{-t}$, $\gamma = 73/45$, $\varphi_1\left(t\right) = t^{1/5}$ and $\varphi_2\left(t\right) = t^{1/3}$. Therefore, all the conditions of Theorem 4.2.1 are satisfied, we conclude that any solution y of (4.30) satisfies the property

$$|y\left(t\right)| \leq C t^{\alpha-1} \text{ and } \left|D_0^{\beta} y\left(t\right)\right| < C t^{\alpha-\beta-1}, \ \alpha = 1/2, \ \beta = 1/4, \ t > 0.$$

Example 4.2.2 *Consider the problem*

$$\begin{cases} D_0^{1/2} y\left(t\right) = t^2 e^{-2t} \left(\cos y\right) \left(y\left(t\right)\right)^{1/3} + t^3 e^{-4t} \left(\sin t^3\right) \left(D_0^{1/4} y\left(t\right)\right)^{1/3}, \ q > 4, \ t > 0, \\ I^{1-1/2} y\left(t\right)|_{t=0} = b. \end{cases} \tag{4.31}$$

We can rewrite (4.31) as follows

$$D_0^{1/2}y\left(t\right)=t^{2+(1/2-1)/3}e^{-t}e^{-t}\left(\cos y\right)\left(t^{1-1/2}y\left(t\right)\right)^{1/3}$$

$$+t^{3+(1/2-1/4-1)/3}e^{-2t}e^{-2t}\left(\sin t^3\right)\left(t^{1-(1/2-1/4)}D_0^{1/4}y\left(t\right)\right)^{1/3}$$

$$=t^{\frac{11}{6}}e^{-t}e^{-t}\left(\cos y\right)\left(t^{1-1/2}y\left(t\right)\right)^{\frac{1}{3}}+t^{\frac{33}{12}}e^{-2t}e^{-2t}\left(\sin t^3\right)\left(t^{1-(1/2-1/4)}D_0^{1/4}y\left(t\right)\right)^{\frac{1}{3}},\ t>0.$$

Let $h_1\left(t\right)=e^{-t}$, $h_2\left(t\right)=e^{-2t}$, $\gamma=11/6$ and $\varphi_1\left(t\right)=\varphi_2\left(t\right)=t^{1/3}$. Therefore all conditions of Theorem 4.2.2 are satisfied so any solution y of (4.31) satisfies the property

$$\left|y\left(t\right)\right|\leq Ct^{\alpha-1}\text{ and }\left|D_0^{\beta}y\left(t\right)\right|<Ct^{\alpha-\beta-1},\ \alpha=1/2,\ \beta=1/4,\ t>0.$$

4.3 Boundedness of Caputo fractional differential equation

In this section, we consider the following fractional differential problem

$$\begin{cases} {}^{C}D_0^{\alpha}y\left(t\right)=f\left(t,y\left(t\right),{}^{C}D_0^{\beta}y\left(t\right)\right),\ 0\leq\beta<\alpha<1,\ t>0, \\ y\left(t\right)|_{t=0}=b, \end{cases}\tag{4.32}$$

in the space

$$C^{\alpha}[0,\infty)=\left\{y\in AC[0,\infty):\ {}^{C}D_0^{\alpha}y\in C[0,\infty)\right\},\tag{4.33}$$

where ${}^{C}D_{0}^{\sigma}$ is Caputo fractional derivative of order $\sigma > 0$.

In the sequel, we consider the following assumption on f :

(C1) The function $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is such that $f(., u(.), v(.)) \in C[0,\infty)$ for any u, $v \in C[0,\infty)$.

(C2) There exist continuous functions h, φ_1, $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|f(t,u,v)| \leq t^{\gamma} h(t) \varphi_1(|u(t)|) \varphi_2(|v(t)|), \ t > 0, \tag{4.34}$$

where $h \in L_q(0,\infty)$ for some $q > \frac{1}{\alpha-\beta}$, $\gamma = \frac{1}{q} - \alpha$, and φ_i, $i = 1, 2$, are nondecreasing functions.

In the next result we establish a useful inequality enjoyed by solutions of Problem (4.32).

Lemma 4.3.1 *Assume that $y \in AC[0,\infty)$ is a solution of (4.32) and f satisfies **(C1)** and **(C2)**. Then, we have*

$$\max\left\{|y(t)|, \left|{}^{C}D_{0}^{\beta} y(t)\right|\right\} \leq z(t), \ t \geq t_0 > 0, \tag{4.35}$$

where

$$z(t) = |b| + K_1 \left(\int_0^t h^q(s) \varphi_1^q(|y(s)|) \varphi_2^q\left(\left|{}^{C}D_{0}^{\beta} y(s)\right|\right) ds\right)^{\frac{1}{q}}, \ t > 0, \tag{4.36}$$

and

$$K_1 = \max\left\{ \frac{K^{1/p}_{p(\alpha-1)+1,p\gamma}}{\Gamma(\alpha)}, \frac{K^{1/p}_{p(\alpha-\beta-1)+1,p\gamma}}{\Gamma(\alpha-\beta)\, t_0^{\beta}} \right\},\ K_{\alpha,\beta} = \frac{\Gamma(\alpha)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},\ p+q=pq.$$

Proof. Applying I_0^{α} to (4.32) and using Lemma 2.3.1, we see that

$$y(t) = b + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), {}^{C}D_0^{\beta} y(s)\right) ds,\ t>0. \tag{4.37}$$

From the inequality (4.34), we obtain

$$|y(t)| \le |b| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\gamma} h(s)\, \varphi_1(|y(s)|)\, \varphi_2\left(\left|{}^{C}D_0^{\beta} y(s)\right|\right) ds,\ t>0. \tag{4.38}$$

It follows from the hypothesis $\beta<\alpha,\ q>\frac{1}{\alpha-\beta}$ and $\gamma=\frac{1}{q}-\alpha$ that $p(\alpha-1)+1 \ge p(\alpha-\beta-1)+1>0$ and $p\gamma+1 = p\left(\frac{1}{q}-\alpha\right)+1 = p(1-\alpha)>0$. Therefore, we can apply Lemma 4.1.2, to get

$$\begin{aligned} |y(t)| &\le |b| + \frac{1}{\Gamma(\alpha)} K^{1/p}_{p(\alpha-1)+1,p\gamma} t^{\alpha+\gamma-1/q} \left(\int_0^t h^q(s)\, \varphi_1^q(|y(s)|)\, \varphi_2^q\left(\left|{}^{C}D_0^{\beta} y(s)\right|\right) ds \right)^{\frac{1}{q}} \\ &\le |b| + K_1 \left(\int_0^t h^q(s)\, \varphi_1^q(|y(s)|)\, \varphi_2^q\left(\left|{}^{C}D_0^{\beta} y(s)\right|\right) ds \right)^{\frac{1}{q}},\ t>0. \end{aligned} \tag{4.39}$$

Also, from Lemma 3.1.3, we deduce

$${}^{C}D_0^{\beta} y(t) = I_0^{\alpha-\beta}\ {}^{C}D^{\alpha} y(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}\ {}^{C}D_0^{\alpha} y(s)\, ds$$

$$= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f\left(s, y(s), {}^C D_0^\beta y(s)\right) ds, \ t > 0. \tag{4.40}$$

By using the inequality (4.34), we are lead to

$$\left|{}^C D_0^\beta y(t)\right| \leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^\gamma h(s) \varphi_1(|y(s)|) \varphi_2\left(\left|{}^C D_0^\beta y(s)\right|\right) ds, \ t > 0.$$

Again, in view of Lemma 4.1.2, we obtain

$$\begin{aligned}
\left|{}^C D_0^\beta y(t)\right| &\leq \frac{K_{p(\alpha-\beta-1)+1, p\gamma}^{1/p}}{\Gamma(\alpha-\beta)} t^{\alpha-\beta+\gamma-1/q} \left(\int_0^t h^q(s) \varphi_1^q(|y(s)|) \varphi_2^q\left(\left|{}^C D_0^\beta y(s)\right|\right) ds\right)^{\frac{1}{q}} \\
&\leq \frac{K_{p(\alpha-\beta-1)+1, p\gamma}^{1/p}}{\Gamma(\alpha-\beta)} t^{-\beta} \left(\int_0^t h^q(s) \varphi_1^q(|y(s)|) \varphi_2^q\left(\left|{}^C D_0^\beta y(s)\right|\right) ds\right)^{\frac{1}{q}} \\
&\leq K_1 \left(\int_0^t h^q(s) \varphi_1^q(|y(s)|) \varphi_2^q\left(\left|{}^C D_0^\beta y(s)\right|\right) ds\right)^{\frac{1}{q}}, \ t \geq t_0 > 0.
\end{aligned} \tag{4.41}$$

The relation (4.35) is an immediate consequence of (4.36), (4.39) and (4.41). ▮

Theorem 4.3.1 *Suppose that f satisfies **(C1)** and **(C2)**. Then, for any solution $y \in AC[0,\infty)$ of Problem (4.32), there exists a positive constant C such that*

$$|y(t)| \leq C \ \text{and} \ \left|{}^C D_0^\beta y(t)\right| < C, \ t > 0,$$

provided that

$$\int_{x_0}^\infty \frac{ds}{\varphi_1^q(s^{1/q}) \varphi_2^q(s^{1/q})} = \infty, \ x_0 > 0.$$

Proof. A simple application of Lemma 4.3.1 gives

$$\varphi_1\left(\left|y\left(t\right)\right|\right) \leq \varphi_1\left(z\left(t\right)\right) \text{ and } \varphi_2\left(\left|{}^{C}D_0^{\beta}y\left(t\right)\right|\right) \leq \varphi_2\left(z\left(t\right)\right), \ t>0. \tag{4.42}$$

From (4.36) and (4.42), we obtain

$$z\left(t\right) \leq \left|b\right| + K_1\left(\int_0^t h^q\left(s\right)\varphi_1^q\left(z\left(s\right)\right)\varphi_2^q\left(z\left(s\right)\right)ds\right)^{\frac{1}{q}}, \ t>0. \tag{4.43}$$

Therefore Lemma 4.1.4 implies

$$z\left(t\right) \leq \left[G^{-1}\left(G\left(2^{q-1}K_1\right) + 2^{q-1}K_2\int_0^t h^q\left(s\right)ds\right)\right]^{1/q} < \infty,$$

because $h \in L_q\left(0,\infty\right)$ and the result follows. ∎

Example 4.3.1 *Consider the problem*

$$\begin{cases} {}^{C}D_0^{2/3}y\left(t\right) = t^{1/q-2/3}e^{-\lambda t}\left(y\left(t\right)\right)^{3/5}\left({}^{C}D_0^{1/3}y\left(t\right)\right)^{1/3}\left(\cos\left({}^{C}D_0^{1/3}y\right)\right) \ t>0, \\ y\left(0\right) = b, \ q>3, \ \lambda>0. \end{cases} \tag{4.44}$$

Let $h\left(t\right) = e^{-\lambda t}$, $\gamma = 1/q - 2/3$, $\varphi_1\left(t\right) = t^{3/5}$ and $\varphi_2\left(t\right) = t^{1/3}$. Then $h \in L_q\left(0,\infty\right)$ and

$$\int_{x_0}^{\infty}\frac{ds}{\varphi_1^q\left(s^{\frac{1}{q}}\right)\varphi_2^q\left(s^{\frac{1}{q}}\right)} = \int_{x_0}^{\infty}\frac{ds}{s^{3/5}s^{1/3}} = \int_{x_0}^{\infty}\frac{ds}{s^{14/15}} = \infty.$$

Therefore, all the conditions of Theorem 4.3.1 are satisfied, we conclude that any solution y of (4.44) satisfies the property

$$|y\,(t)| \leq C \text{ and } \left| {}^{C}D_{0}^{\beta} y\,(t) \right| < C,\ \ \alpha = 2/3,\ \ \beta = 1/3,\ t > 0.$$

CHAPTER 5

GLOBAL NON-EXISTENCE FOR FRACTIONALLY DAMPED FRACTIONAL DIFFERENTIAL PROBLEMS WITH POWER TYPE SOURCE TERM

In this chapter, we are concerned with a fractional differential equation containing a lower order fractional derivative

$$D_0^{\alpha} y(t) + D_0^{\beta} y(t) = f(t, y(t)), \ 0 < \beta \leq \alpha \leq 1, \, t > 0, \tag{5.1}$$

where D_0^{σ} is either the Riemann-Liouville derivative or the Caputo derivative.

A non-existence of non-trivial global solutions result is proved in an appropriate space by means of the test-function method. The range of blow up is found to depend only on the lower order derivative. This is in line with the well-known fact for an internally weakly damped wave equation that solutions will converge to solutions of the parabolic part.

Here, we would like to investigate the case where a lower order fractional derivative is present in the equation (or inequality). Then, we show that fractional derivatives of lower order have a strong influence on the character of the solutions. Our method of proof relies on a suitable choice of the test function

In this chapter, we use the following test functions:

1) The function $\varphi(t) \in C^1([0, \infty))$ satisfies: $\varphi(t) \geq 0$, non-increasing and such that

$$\varphi(t) := \begin{cases} 1, & t \in [0, T/2], \\ 0, & t \in [T, \infty), \end{cases} \tag{5.2}$$

for some $T > 0$.

2)

$$\varphi(t) = \begin{cases} T^{-\lambda}(T-t)^{\lambda}, \ 0 \le t \le T, \ \lambda > 0, \\ 0, \ t > T. \end{cases} \tag{5.3}$$

5.1 Preliminaries

In this section, we prove some lemmas, properties which will be used in our result later.

Lemma 5.1.1 *Let $0 \le \gamma < 1$ and $f \in C_\gamma[a,b]$. Then*

$$I_a^\alpha f(a) = \lim_{t \to a} I_a^\alpha f(t) = 0, \quad 0 \le \gamma < \alpha,$$

and

$$I_{b^-}^\alpha f(b) = \lim_{t \to b} I_{b^-}^\alpha f(t) = 0, \quad 0 \le \gamma < \alpha.$$

Proof. Note that $(t-a)^\gamma f(t)$ is a continuous on $[a,b]$, because $f \in C_\gamma[a,b]$, and on $[a,b]$ we have

$$|(t-a)^\gamma f(t)| < M, \ M > 0.$$

Therefore

$$|I_a^\alpha f(t)| < M\left[I_a^\alpha (s-a)^{-\gamma}\right](t) = M\frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}(t-a)^{\alpha-\gamma}, \ t > a.$$

As $\alpha > \gamma$ we see that

$$I_a^{\alpha} f(a) = \lim_{t \to a} I_a^{\alpha} f(t) = 0, \quad 0 \le \gamma < \alpha.$$

The second part is proved similarly. ∎

The test functions (5.2) and (5.3) enjoy the following properties.

Lemma 5.1.2 *Let φ be as in (5.2), then*

$$I(T) = \int_{T/2}^{T} \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^p} \right)^m (t)\, dt \le K_{\alpha,m} T^{1-\alpha m}, \quad 0 < \alpha < 1,\ T,\ p,\ m > 0 \tag{5.4}$$

where

$$K_{\alpha,m} = \frac{K_1^m}{2^{m(1-\alpha)+1} \Gamma^m (2-\alpha) \left[m(1-\alpha) + 1 \right]}, \tag{5.5}$$

and K_1 is a bound for $\frac{|\varphi'(r)|}{\varphi(r)^p}$.

Proof. Using definition of Riemann-Liouville right-sided fractional integral (Definition 2.2.2), we see that

$$I(T) = \int_{T/2}^{T} \left(\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} \frac{|\varphi'(s)|}{\varphi(s)^p} ds \right)^m dt. \tag{5.6}$$

The change of variable $\sigma T = t$ in (5.6) yields

$$I(T) = \int_{1/2}^{1} \left(\frac{1}{\Gamma(1-\alpha)} \int_{\sigma T}^{T} (s - \sigma T)^{-\alpha} \frac{|\varphi'(s)|}{\varphi(s)^p} ds \right)^m T d\sigma. \tag{5.7}$$

Another change of variable $s = rT$ in (5.7) gives

$$I(T) = \int_{1/2}^{1} \left(\frac{1}{\Gamma(1-\alpha)} \int_{\sigma}^{1} (rT - \sigma T)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^p} dr \right)^m T d\sigma$$

$$= \frac{T^{1-\alpha m}}{\Gamma^m(1-\alpha)} \int_{1/2}^{1} \left(\int_{\sigma}^{1} (r-\sigma)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^p} dr \right)^m d\sigma. \tag{5.8}$$

We may assume without loss of generality that

$$\frac{|\varphi'(r)|}{\varphi(r)^p} \leq K_1,$$

for some positive constant K_1, for otherwise we consider $\varphi^\lambda(r)$ with some sufficiently large λ. Therefore from (5.8) we get

$$I(T) \leq \frac{K_1^m T^{1-\alpha m}}{\Gamma^m(1-\alpha)} \int_{1/2}^{1} \left(\int_{\sigma}^{1} (r-\sigma)^{-\alpha} dr \right)^m d\sigma = \frac{K_1^m T^{1-\alpha m}}{\Gamma^m(2-\alpha)} \int_{1/2}^{1} (1-\sigma)^{m(1-\alpha)} d\sigma$$

$$= \frac{K_1^m}{2^{m(1-\alpha)+1}\Gamma^m(2-\alpha)\left[m(1-\alpha)+1\right]} T^{1-\alpha m}.$$

Therefore

$$I(T) \leq K_{\alpha,m} T^{1-\alpha m}.$$

∎

Lemma 5.1.3 *Let* $\alpha > 0$, $n = -[-\alpha]$, *and* φ *be as in (5.3) with* $\lambda > \alpha - 1$. *If* $f \in AC^n[0,T]$, $T > 0$, *then*

$$\int_0^T \varphi(t)\, {}^C D_0^\alpha f(t)\, dt = \int_0^T f(t)\, D_T^\alpha \varphi(t)\, dt - \sum_{i=0}^{n-1} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+i+2)} T^{-\alpha+i+1} f^{(i)}(0).$$

Proof. In view of Property 2.2.1 and the definition of the function φ, we have

$$D_T^{\alpha-i-1}\varphi(t) = T^{-\lambda} D_T^{\alpha-i-1}(T-t)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+i+2)} T^{-\lambda}(T-t)^{\lambda-\alpha+i+1},\ 0 \le t \le T,$$

then

$$D_T^{\alpha-i-1}\varphi(0) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+i+2)} T^{-\alpha+i+1},$$

$$D_T^{\alpha-i-1}\varphi(T) = 0,\ i = 0, 1, 2..., n-1.$$

Now, the result follows from Lemma 2.4.2 ∎

Lemma 5.1.4 *Let φ be as in (5.3) with $\lambda > \beta - 1$ and β, $\alpha \ge 0$. Then*

$$I_T^{\alpha} D_T^{\beta}\varphi(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)} T^{-\lambda}(T-t)^{\alpha+\lambda-\beta},\ 0 \le t < T.$$

Proof. By using Property 2.2.1, we have

$$D_T^{\beta}\varphi(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)} T^{-\lambda}(T-t)^{\lambda-\beta},\ 0 \le t < T.$$

Again by Property 2.2.1, we get

$$I_T^{\alpha} D_T^{\beta}\varphi(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)} T^{-\lambda} I_T^{\alpha}(T-t)^{\lambda-\beta}$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)} T^{-\lambda}(T-t)^{\alpha+\lambda-\beta},\ 0 \le t < T.$$

∎

Lemma 5.1.5 *Let φ be as in (5.3) with $\lambda > \max\{p(\beta-\alpha)-1, \beta-1\}$, β, $\alpha \geq 0$ and $p > 1$. Then*

$$\int_0^T t^{\gamma(1-p)}\varphi^{1-p}(t)\left[I_T^{\alpha}D_T^{\beta}\varphi(t)\right]^p dt = C_{\lambda,\alpha,\beta}^{\gamma,p}T^{\gamma(1-p)+p(\alpha-\beta)+1},\ \gamma(1-p)+1>0,$$

where

$$C_{\lambda,\alpha,\beta}^{\gamma,p} = \left[\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)}\right]^p \frac{\Gamma(\gamma(1-p)+1)\Gamma(p(\alpha-\beta)+\lambda+1)}{\Gamma(\gamma(1-p)+p(\alpha-\beta)+\lambda+2)}.$$

Proof. From (5.3) and Lemma 5.1.4, we have

$$\varphi^{1-p}(t)\left[I_T^{\alpha}D_T^{\beta}\varphi(t)\right]^p = \left[T^{-\lambda}(T-t)^{\lambda}\right]^{1-p}\left[\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)}\right]^p T^{-p\lambda}(T-t)^{p(\alpha+\lambda-\beta)}$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)}\right]^p T^{-\lambda}(T-t)^{p(\alpha-\beta)+\lambda},\ 0\leq t<T.$$

Therefore

$$\int_0^T t^{\gamma(1-p)}\varphi^{1-p}(t)\left[I_T^{\alpha}D_T^{\beta}\varphi(t)\right]^p dt$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)}\right]^p T^{-\lambda}\int_0^T t^{\gamma(1-p)}(T-t)^{p(\alpha-\beta)+\lambda}dt.$$

Let $s = t/T$. Then

$$\int_0^T t^{\gamma(1-p)}\varphi^{1-p}(t)\left[I_T^{\alpha}D_T^{\beta}\varphi(t)\right]^p dt$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)}\right]^p T^{\gamma(1-p)+p(\alpha-\beta)+1}\int_0^1 s^{\gamma(1-p)}(1-s)^{p(\alpha-\beta)+\lambda}ds$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda-\beta+1)}\right]^p \frac{\Gamma(\gamma(1-p)+1)\Gamma(p(\alpha-\beta)+\lambda+1)}{\Gamma(\gamma(1-p)+p(\alpha-\beta)+\lambda+2)}T^{\gamma(1-p)+p(\alpha-\beta)+1}.$$

Lemma 5.1.6 *Let φ be as in (5.3) with $\lambda > p\beta - 1$, $\beta > 0$ and $p > 1$. Then*

$$\int_0^T t^{\gamma(1-p)}\varphi^{1-p}(t)\left[D_T^\beta \varphi(t)\right]^p dt = C_{\lambda,\beta}^{\gamma,p} T^{\gamma(1-p)-p\beta+1},\ \gamma(1-p)+1>0,$$

where

$$C_{\lambda,\beta}^{\gamma,p} = \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)}\right]^p \frac{\Gamma(\gamma(1-p)+1)\,\Gamma(\lambda-p\beta+1)}{\Gamma(\gamma(1-p)+\lambda-p\beta+2)}.$$

Proof. From (5.3) and Property 2.2.1, we have

$$\varphi^{1-p}(t)\left[D_T^\beta \varphi(t)\right]^p = \left[T^{-\lambda}(T-t)^\lambda\right]^{1-p}\left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)}\right]^p T^{-p\lambda}(T-t)^{p(\lambda-\beta)}$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)}\right]^p T^{-\lambda}(T-t)^{\lambda-p\beta},\ 0 \le t < T.$$

Therefore

$$\int_0^T t^{\gamma(1-p)}\varphi^{1-p}(t)\left[D_T^\beta \varphi(t)\right]^p dt = \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)}\right]^p T^{-\lambda}\int_0^T t^{\gamma(1-p)}(T-t)^{\lambda-p\beta}\,dt.$$

Let $s = t/T$. Then

$$\int_0^T t^{\gamma(1-p)}\varphi^{1-p}(t)\left[D_T^\beta \varphi(t)\right]^p dt$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)}\right]^p T^{\gamma(1-p)-p\beta+1}\int_0^1 s^{\gamma(1-p)}(1-s)^{\lambda-p\beta}\,ds$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)}\right]^p \frac{\Gamma(\gamma(1-p)+1)\,\Gamma(\lambda-p\beta+1)}{\Gamma(\gamma(1-p)+\lambda-p\beta+2)} T^{\gamma(1-p)-p\beta+1}.$$

5.2 Non-existence of global solutions for Riemann-Liouville fractional derivatives

A nonexistence result of non-trivial global solutions for the problem (5.1) will be considered when $f(t, y(t)) \geq t^{\gamma} |y(t)|^{m}$ for some $m > 1$ and $\gamma \in \mathbb{R}$ in the space $C^{\alpha}_{1-\alpha}[0, \infty)$ defined in (4.16). That is we consider the problem:

$$\begin{cases} D_0^{\alpha} y(t) + D_0^{\beta} y(t) \geq t^{\gamma} |y(t)|^{m}, \ t > 0, \\ I_0^{1-\alpha} y(t)\,|_{t=0} = b, \end{cases} \tag{5.9}$$

where $0 < \beta \leq \alpha \leq 1$ and show that there are no solutions exist for all time for certain values of γ and m. In particular, we find the range of values of m for which solutions do not exist globally.

Theorem 5.2.1 *Assume that $\gamma > -\beta$ and $1 < m \leq \frac{\gamma+1}{1-\beta}$. Then, Problem (5.9) does not admit global nontrivial solutions in $C^{\alpha}_{1-\alpha}$, when $b \geq 0$.*

Proof. Assume, on the contrary, that a nontrivial solution y exists for all time $t > 0$. Let φ be as in (5.2). Multiplying the inequality in (5.9) by $\varphi(t)$ and integrating over $(0, T)$, we get

$$I_1 := \int_0^T t^{\gamma} |y(t)|^{m} \varphi(t)\, dt \leq \int_0^T D_0^{\alpha} y(t)\, \varphi(t)\, dt + \int_0^T D_0^{\beta} y(t)\, \varphi(t)\, dt. \tag{5.10}$$

Let

$$I_2 := \int_0^T \varphi(t)\, D_0^{\alpha} y(t)\, dt,$$

and

$$I_3 := \int_0^T \varphi(t)\, D_0^{\beta} y(t)\, dt.$$

From the definition of $D_0^{\alpha} y$ (Definition 2.2.3), we can write

$$I_2 = \int_0^T \varphi(t)\, \frac{d}{dt} I_0^{1-\alpha} y(t)\, dt.$$

An integration by parts yields

$$I_2 = \left[\varphi(t)\, I_0^{1-\alpha} y(t)\right]_{t=0}^{T} - \int_0^T \varphi'(t)\, I_0^{1-\alpha} y(t)\, dt.$$

Since $\varphi(T) = 0$, $\varphi(0) = 1$ and $I_0^{1-\alpha} y(0) = b$, then

$$I_2 = -b - \int_0^T \varphi'(t)\, I_0^{1-\alpha} y(t)\, dt.$$

As $b \geq 0$, we get

$$I_2 \leq -\int_0^T \varphi'(t)\, I_0^{1-\alpha} y(t)\, dt \leq \int_0^T |\varphi'(t)| \left(I_0^{1-\alpha} |y|\right)(t)\, dt$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^{\alpha}} ds dt. \tag{5.11}$$

Because $\varphi(t)$ is nonincreasing $\varphi(s) \geq \varphi(t)$ for all $t \geq s$, and therefore

$$\frac{1}{\varphi(s)^{1/m}} \leq \frac{1}{\varphi(t)^{1/m}},\ m > 1.$$

Also we have

$$\varphi'(t) = 0,\ t \in [0, T/2].$$

Thus

$$I_2 \le \frac{1}{\Gamma(1-\alpha)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \frac{\varphi(s)^{1/m}}{\varphi(s)^{1/m}} dsdt$$

$$\le \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \varphi(s)^{1/m} dsdt$$

$$\le \frac{1}{\Gamma(1-\alpha)} \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \varphi(s)^{1/m} dsdt$$

$$\le \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} \left(I_0^{1-\alpha} \varphi^{1/m} |y|\right)(t)\, dt.$$

A fractional integration by parts, Lemma 2.4.1, in the last expression yields

$$I_2 \le \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\, \varphi(t)^{1/m} |y(t)|\, dt.$$

Next, we multiply by $t^{\gamma/m} t^{-\gamma/m}$ inside the integral in the right hand side

$$I_2 \le \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\, \varphi(t)^{1/m} \frac{t^{\gamma/m}}{t^{\gamma/m}} |y(t)|\, dt.$$

For $\gamma < 0$ we have $t^{-\gamma/m} < T^{-\gamma/m}$ (because $t < T$) and for $\gamma > 0$ we get $t^{-\gamma/m} < 2^{\gamma/m} T^{-\gamma/m}$ (because $T/2 < t$): that is

$$t^{-\gamma/m} < \max\left\{1, 2^{\gamma/m}\right\} T^{-\gamma/m}.$$

Then

$$I_2 \leq \max\left\{1, 2^{\gamma/m}\right\} T^{-\gamma/m} \int_{T/2}^{T} \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)(t)\, t^{\gamma/m} \varphi(t)^{1/m} |y(t)|\, dt. \tag{5.12}$$

By Hölder's inequality, it is clear that

$$I_2 \leq \max\left\{1, 2^{\gamma/m}\right\} T^{-\gamma/m} \left(\int_{T/2}^{T} t^{\gamma} \varphi(t) |y(t)|^m dt \right)^{\frac{1}{m}} \left(\int_{T/2}^{T} \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}} \right)^{m'} (t)\, dt \right)^{\frac{1}{m'}}.$$

Lemma 5.1.2 implies that

$$I_2 \leq \max\left\{1, 2^{\gamma/m}\right\} T^{-\gamma/m} \left(\int_{T/2}^{T} t^{\gamma} \varphi(t) |y(t)|^m dt \right)^{\frac{1}{m}} \left(K_{\alpha,m'} T^{1-\alpha m'} \right)^{\frac{1}{m'}}, \tag{5.13}$$

where $K_{\alpha,m'}$ is the constant appearing in Lemma 5.1.2 corresponding to the present exponents. Therefore from (5.13) we have the estimate

$$I_2 \leq \max\left\{1, 2^{\gamma/m}\right\} K_{\alpha,m'}^{\frac{1}{m'}} T^{1/m'-\alpha-\gamma/m} I_1^{\frac{1}{m}}. \tag{5.14}$$

Now, we turn to I_3. First, notice that $y \in C_{1-\alpha}[0,T]$ and $1-\alpha < 1-\beta$, then by Lemma 5.1.1 we have

$$I_0^{1-\beta} y(0) = \lim_{t \to 0} I_0^{1-\beta} y(t) = 0.$$

An integration by parts in

$$I_3 = \int_0^T \varphi(t)\, D_0^\beta y(t)\, dt = \int_0^T \varphi(t) \frac{d}{dt} I_0^{1-\beta} y(t)\, dt$$

gives

$$I_3 = \left[\varphi(t)\, I_0^{1-\beta} y(t)\right]_{t=0}^{T} - \int_0^T \varphi'(t)\, I_0^{1-\beta} y(t)\, dt.$$

Therefore

$$I_3 = -\int_0^T \varphi'(t)\, I_0^{1-\beta} y(t)\, dt \leq \int_0^T |\varphi'(t)| \left(I_0^{1-\beta} |y|\right)(t)\, dt$$

$$\leq \frac{1}{\Gamma(1-\beta)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\beta} ds dt.$$

Replacing α by β in the argument above allows us to write

$$I_3 \leq \max\left\{1, 2^{\gamma/m}\right\} T^{-\gamma/m} \int_{T/2}^T \left(I_{T-}^{1-\beta} \frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\, t^{\gamma/m} \varphi(t)^{1/m} |y(t)|\, dt, \tag{5.15}$$

or simply

$$I_3 \leq K_{\beta,m'}^{\frac{1}{m'}} \max\left\{1, 2^{\gamma/m}\right\} T^{1/m' - \beta - \gamma/m} I_1^{\frac{1}{m}}. \tag{5.16}$$

From (5.10), (5.14) and (5.16), we have

$$I_1 \leq \max\left\{1, 2^{\gamma/m}\right\} K_{\alpha,m'}^{\frac{1}{m'}} T^{1/m' - \alpha - \gamma/m} I_1^{\frac{1}{m}} + K_{\beta,m'}^{\frac{1}{m'}} \max\left\{1, 2^{\gamma/m}\right\} T^{1/m' - \beta - \gamma/m} I_1^{\frac{1}{m}}$$

$$\leq \max\left\{K_{\alpha,m'}^{\frac{1}{m'}}, K_{\beta,m'}^{\frac{1}{m'}}\right\} \max\left\{1, 2^{\gamma/m}\right\} \left(T^{1/m' - \alpha - \gamma/m} + T^{1/m' - \beta - \gamma/m}\right) I_1^{\frac{1}{m}}.$$

Therefore

$$I_1^{\frac{1}{m'}} \le K_2 \left(T^{1/m'-\alpha-\gamma/m} + T^{1/m'-\beta-\gamma/m} \right), \tag{5.17}$$

with

$$K_2 := \max\left\{ K_{\alpha,m'}^{\frac{1}{m'}}, K_{\beta,m'}^{\frac{1}{m'}} \right\} \max\left\{1, 2^{\gamma/m}\right\}.$$

Raising both sides of (5.17) to the power m' and using Lemma 2.4.4, we get

$$I_1 \le K_3 \left(T^{1-\alpha m'-\gamma m'/m} + T^{1-\beta m'-\gamma m'/m} \right), \tag{5.18}$$

with

$$K_3 = 2^{1-m'} K_2^{m'}.$$

If $m < \frac{\gamma+1}{1-\beta}$ we see that $1 - \beta m' - \gamma m'/m < 0$, $1 - \alpha m' - \gamma m'/m < 0$, and consequently $T^{1-\beta m'-\gamma m'/m} \to 0$ and $T^{1-\alpha m'-\gamma m'/m} \to 0$ as $T \to \infty$. Then, from (5.18), we obtain

$$\lim_{T\to\infty} \int_0^T t^\gamma \left|y\left(t\right)\right|^m \varphi\left(t\right) dt = 0.$$

This implies that $y = 0$. We arrive at a contradiction.

In the case $m = \frac{\gamma+1}{1-\beta}$ we find $1 - \beta m' - \gamma m'/m = 0$, $1 - \alpha m' - \gamma m'/m \le 0$, and the relation (5.18) ensures that

$$\lim_{T\to\infty} \int_0^T t^\gamma \left|y\left(t\right)\right|^m \varphi\left(t\right) dt \le K_4. \tag{5.19}$$

Further, in view of (5.10), (5.12) and (5.15), we see that

$$I_1 \leq \max\left\{1, 2^{\gamma/m}\right\} T^{\frac{-\gamma}{m}} \int_{T/2}^{T} t^{\frac{\gamma}{m}} \varphi\left(t\right)^{\frac{1}{m}} \left|y\left(t\right)\right| \left[\left(I_{T-}^{1-\alpha} \frac{\left|\varphi'\right|}{\varphi^{\frac{1}{m}}}\right)\left(t\right) + \left(I_{T-}^{1-\beta} \frac{\left|\varphi'\right|}{\varphi^{\frac{1}{m}}}\right)\left(t\right)\right] dt.$$

Thanks to Hölder's inequality, it is clear that

$$\begin{aligned} I_1 \leq{}& \max\left\{1, 2^{\gamma/m}\right\} T^{-\gamma/m} \left[\int_{T/2}^{T} t^{\gamma} \varphi\left(t\right) \left|y\left(t\right)\right|^m dt\right]^{\frac{1}{m}} \\ & \times \left\{\int_{T/2}^{T} \left[\left(I_{T-}^{1-\alpha} \frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)\left(t\right) + \left(I_{T-}^{1-\beta} \frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)\left(t\right)\right]^{m'} dt\right\}^{\frac{1}{m'}} \\ \leq{}& \max\left\{1, 2^{\gamma/m}\right\} 2^{1/m} T^{-\gamma/m} \left[\int_{T/2}^{T} t^{\gamma} \varphi\left(t\right) \left|y\left(t\right)\right|^m dt\right]^{\frac{1}{m}} \\ & \times \left\{\int_{T/2}^{T} \left[\left(I_{T-}^{1-\alpha} \frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}\left(t\right) + \left(I_{T-}^{1-\beta} \frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}\left(t\right)\right] dt\right\}^{\frac{1}{m'}}. \end{aligned}$$

Therefore, by Lemma 5.1.2, we obtain

$$\begin{aligned} I_1 \leq{}& K_5 T^{-\gamma/m} \left[\int_{T/2}^{T} t^{\gamma} \varphi\left(t\right) \left|y\left(t\right)\right|^m dt\right]^{\frac{1}{m}} \left[K_{\alpha,m'} T^{1-\alpha m'} + K_{\beta,m'} T^{1-\beta m'}\right]^{\frac{1}{m'}} \\ ={}& K_5 \left[\int_{T/2}^{T} t^{\gamma} \varphi\left(t\right) \left|y\left(t\right)\right|^m dt\right]^{\frac{1}{m}} \left[K_{\alpha,m'} T^{1-\alpha m'-\gamma m'/m} + K_{\beta,m'} T^{1-\beta m'-\gamma m'/m}\right]^{\frac{1}{m'}}, \end{aligned}$$

with

$$K_5 = \max\left\{1, 2^{\gamma/m}\right\} 2^{1/m}.$$

Since $m = \frac{\gamma+1}{1-\beta}$, then $1 - \beta m' - \gamma m'/m = 0$ and $1 - \alpha m' - \gamma m'/m \leq 0$. Therefore

$$I_1 \leq K_6 \left[\int_{T/2}^{T} t^{\gamma} \varphi(t) \left| y(t) \right|^m dt \right]^{\frac{1}{m}}$$

for some positive constant K_6, with

$$\lim_{T \to \infty} \int_{T/2}^{T} t^{\gamma} \varphi(t) \left| y(t) \right|^m dt = 0$$

because of the convergence of the integral in (5.19). This is again a contradiction. The proof is complete. ∎

5.3 Non-existence of global solutions for Caputo fractional derivatives

We consider the problem

$$\begin{cases} {}^{C}D_0^{\alpha} y(t) + {}^{C}D_0^{\beta} y(t) \geq t^{\gamma} \left| y(t) \right|^m, \quad 0 < \beta < \alpha, \ m > 1, \quad t > 0 \\ y^{(i)}(0) = b_i, \ i = 0, 1, ..., n-1, \ n = -\left[-\alpha\right], \end{cases} \tag{5.20}$$

where ${}^{C}D_0^{\sigma}$ is the Caputo fractional derivative of order $\sigma > 0$.

Theorem 5.3.1 *Assume that $m(1-\beta) - 1 < \gamma < m - 1$. Then, Problem (5.20) does not admit global nontrivial solutions in $AC^n[0, \infty)$, when $b_i \geq 0$.*

Proof. Assume, on the contrary, that a solution y exists for all time $t > 0$. Let φ be as in (5.3) with $\lambda > \frac{m\alpha}{m-1} - 1$. Multiplying the inequality in (5.20) by φ and integrating over $[0,T]$, we get

$$I=\int_0^T \varphi(t)\, t^{\gamma}\, |y(t)|^m\, dt \le \int_0^T \varphi(t)\, {}^C D_0^{\alpha} y(t)\, dt + \int_0^T \varphi(t)\, {}^C D_0^{\beta} y(t)\, dt. \quad (5.21)$$

By Lemma 5.1.3, we conclude

$$\int_0^T \varphi(t)\, {}^C D_0^{\alpha} y(t)\, dt = \int_0^T y(t)\, D_T^{\alpha}\varphi(t)\, dt - \sum_{i=0}^{n-1} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+i+2)} T^{i+1-\alpha} y^{(i)}(0).$$

$$= \int_0^T y(t)\, D_T^{\alpha}\varphi(t)\, dt - \sum_{i=0}^{n-1} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+i+2)} T^{i+1-\alpha} b_i, \quad (5.22)$$

and

$$\int_0^T \varphi(t)\, {}^C D_0^{\beta} y(t)\, dt = \int_0^T y(t)\, D_T^{\beta}\varphi(t)\, dt - \sum_{i=0}^{n'-1} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+i+2)} T^{1+i-\beta} b_i, \quad (5.23)$$

where $n'=-[-\beta]$.

As $b_i \ge 0$, we deduce from (5.21), (5.22) and (5.23)

$$I=\int_0^T \varphi(t)\, t^{\gamma}\, |y(t)|^m \le \int_0^T y(t)\, D_T^{\alpha}\varphi(t)\, dt + \int_0^T y(t)\, D_T^{\beta}\varphi(t)\, dt.$$

Next, we multiply by $\varphi^{1/m} t^{\gamma/m} \varphi^{-1/m} t^{-\gamma/m}$ inside the integrals in the right hand sides

$$I \le \int_0^T y(t)\, \varphi^{1/m} t^{\gamma/m} \varphi^{-1/m} t^{-\gamma/m} D_T^{\alpha}\varphi(t)\, dt$$
$$+ \int_0^T y(t)\, \varphi^{1/m} t^{\gamma/m} \varphi^{-1/m} t^{-\gamma/m} D_T^{\beta}\varphi(t)\, dt. \quad (5.24)$$

By applying Hölder's inequality on the integrals in the right hand sides of (5.24), we obtain

$$I \leq \left(\int_0^T \varphi(t)\, t^{\gamma} |y(t)|^m \, dt \right)^{1/m} \left[\int_0^T \varphi^{-m'/m} t^{-\gamma m'/m} \left| D_T^{\alpha} \varphi(t) \right|^{m'} dt \right]^{1/m'}$$

$$+ \left(\int_0^T \varphi(t)\, t^{\gamma} |y(t)|^m \, dt \right)^{1/m} \left[\int_0^T \varphi^{-m'/m} t^{-\gamma m'/m} \left| D_T^{\beta} \varphi(t) \right|^{m'} dt \right]^{1/m'},$$

or

$$I^{1/m'} \leq \left[\int_0^T \varphi^{-m'/m} t^{-\gamma m'/m} \left| D_T^{\alpha} \varphi(t) \right|^{m'} dt \right]^{1/m'}$$

$$+ \left[\int_0^T \varphi^{-m'/m} t^{-\gamma m'/m} \left| D_T^{\beta} \varphi(t) \right|^{m'} dt \right]^{1/m'}. \tag{5.25}$$

Now by Lemma 5.1.6, we find

$$I^{1/m'} \leq \left[C_{\lambda,\alpha}^{\gamma,m'} T^{\gamma(1-m')-\alpha m'+1} \right]^{1/m'} + \left[C_{\lambda,\beta}^{\gamma,m'} T^{\gamma(1-m')-\beta m'+1} \right]^{1/m'}. \tag{5.26}$$

Raising both sides of (5.26) to the power m' we have

$$I \leq C \left[T^{\gamma(1-m')-\alpha m'+1} + T^{\gamma(1-m')-\beta m'+1} \right],$$

where

$$C = 2^{m'-1} \max \left\{ C_{\lambda,\alpha}^{\gamma,m'}, C_{\lambda,\beta}^{\gamma,m'} \right\}.$$

If $\gamma > m(1-\beta) - 1$ we see that $\gamma(1-m') - \beta m' + 1 < 0$ and $\gamma(1-m') - \alpha m' + 1 <$

0, and consequently $T^{\gamma(1-m')-\alpha m'+1}$, $T^{\gamma(1-m')-\beta m'+1} \to 0$ as $T \to \infty$. Therefore

$$\lim_{T\to\infty} \int_0^T \varphi(t)\, t^{\gamma} \left|y(t)\right|^m = 0.$$

This is a contradiction. ∎

CHAPTER 6

ASYMPTOTIC BEHAVIOR FOR A SYSTEM

In this chapter, we will discuss the asymptotic behaviour of global solutions for the following system of equations

$$\begin{cases} D_0^{\alpha_1} y_1(t) = g_1\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right) \\ D_0^{\alpha_2} y_2(t) = g_2\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right) \end{cases}, \ 0 \le \beta_1, \beta_2 < \min\{\alpha_1, \alpha_2\} < 2, \quad t > 0 \tag{6.1}$$

where the derivative in the system could be the Riemann-Liouville derivative or the Caputo derivative.

We shall establish some conditions ensuring a similar behavior to power functions as time goes to infinity for solutions of system (6.1).

6.1 Preliminaries

First, we need to prove the following results.

Theorem 6.1.1 *Let* $u,\ v,\ f_i \in C\left[(0,\infty), \mathbb{R}_+\right],\ i = 1,2,3,4$ *and* $c_j > 0,\ j = 1,2.$ *If*

$$u(t) \le c_1 + \int_0^t f_1(s)\, u(s)\, ds + \int_0^t f_2(s)\, v(s)\, ds, \ t > 0, \tag{6.2}$$

$$v(t) \le c_2 + \int_0^t f_3(s)\, u(s)\, ds + \int_0^t f_4(s)\, v(s)\, ds, \ t > 0, \tag{6.3}$$

then

$$u(t) \le \left[c_1 + c_2 \int_0^t f_2(s) \exp\left(\int_0^s f_3(\tau)\, d\tau\right) ds\right]$$

$$\times \exp\left\{\int_0^t f_1(s)\, ds + \int_0^t f_2(s) \exp\left(\int_0^s f_4(\tau)\, d\tau\right)\right.$$

$$\times \left[\int_0^s f_3(\tau)\exp\left(\int_0^\tau f_1(\sigma)\,d\sigma\right)d\tau\right]ds\Bigg\},\ t>0,$$

and

$$v(t) \leq \left[c_2 + c_1\int_0^t f_3(s)\exp\left(\int_0^s f_1(\tau)\,d\tau\right)ds\right]$$
$$\times \exp\left\{\int_0^t f_4(s)\,ds + \int_0^t f_3(s)\exp\left(\int_0^s f_1(\tau)\,d\tau\right)\right.$$
$$\left.\times \left[\int_0^s f_2(\tau)\exp\left(\int_0^\tau f_4(\sigma)\,d\sigma\right)d\tau\right]ds\right\},\ t>0.$$

Proof. Let

$$h(t) = c_1 + \int_0^t f_2(s)\,v(s)\,ds,\ t>0. \tag{6.4}$$

Then (6.2) becomes

$$u(t) \leq h(t) + \int_0^t f_1(s)\,u(s)\,ds,\ t>0.$$

Clearly that h is a continuous, positive and nondecreasing function defined for all $t>0$, therefore by Theorem 2.4.3, we see that

$$u(t) \leq h(t)\exp\left(\int_0^t f_1(s)\,ds\right),\ t>0. \tag{6.5}$$

Substituting (6.5) into (6.3), we obtain

$$v(t) \leq c_2 + \int_0^t f_3(s)\,h(s)\exp\left(\int_0^s f_1(\tau)\,d\tau\right)ds + \int_0^t f_4(s)\,v(s)\,ds,\ t>0. \tag{6.6}$$

Let us insert the expression (6.4) into (6.6), we find

$$v(t) \leq c_2 + c_1 \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) ds$$

$$+ \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) \left[\int_0^s f_2(\tau)\, v(\tau)\, d\tau\right] ds + \int_0^t f_4(s)\, v(s)\, ds,\ t > 0. \tag{6.7}$$

Put

$$g(t) = c_2 + c_1 \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) ds$$

$$+ \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) \left[\int_0^s f_2(\tau)\, v(\tau)\, d\tau\right] ds,\ t > 0. \tag{6.8}$$

From (6.7) and (6.8), we deduce

$$v(t) \leq g(t) + \int_0^t f_4(s)\, v(s)\, ds,\ t > 0.$$

Clear that g is a continuous, positive and nondecreasing function defined for all $t > 0$, therefore, by Theorem 2.4.3, we get

$$v(t) \leq g(t) \exp\left(\int_0^t f_4(s)\, ds\right),\ t > 0. \tag{6.9}$$

Substituting (6.9) into (6.8), we have

$$g(t) \leq c_2 + c_1 \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) ds$$

$$+ \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) \left[\int_0^s f_2(\tau)\, g(\tau) \exp\left(\int_0^\tau f_4(\sigma)\, d\sigma\right) d\tau\right] ds,\ t > 0.$$

Since g is nondecreasing, then

$$g(t) \leq c_2 + c_1 \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) ds$$

$$+ \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) \left[\int_0^s f_2(\tau) \exp\left(\int_0^\tau f_4(\sigma)\, d\sigma\right) d\tau\right] g(s)\, ds, \ t > 0. \tag{6.10}$$

Set

$$k(t) = c_2 + c_1 \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) ds, \ t > 0. \tag{6.11}$$

From (6.10) and (6.11), we conclude

$$g(t) \leq k(t)$$

$$+ \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) \left[\int_0^s f_2(\tau) \exp\left(\int_0^\tau f_4(\sigma)\, d\sigma\right) d\tau\right] g(s)\, ds, \ t > 0.$$

Applying Theorem 2.4.3, we infer that

$$g(t) \leq k(t)$$

$$\times \exp\left(\int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) \left[\int_0^s f_2(\tau) \exp\left(\int_0^\tau f_4(\sigma)\, d\sigma\right) d\tau\right] ds\right), \ t > 0. \tag{6.12}$$

By (6.9), (6.11) and (6.12), we obtain

$$v(t) \leq \left[c_2 + c_1 \int_0^t f_3(s) \exp\left(\int_0^s f_1(\tau)\, d\tau\right) ds\right]$$

$$\times \exp\left\{\int_0^t f_4(s)\,ds + \int_0^t f_3(s)\exp\left(\int_0^s f_1(\tau)\,d\tau\right)\right.$$

$$\left.\times\left[\int_0^s f_2(\tau)\exp\left(\int_0^\tau f_4(\sigma)\,d\sigma\right)d\tau\right]ds\right\},\ t>0.$$

By the same way we can prove

$$u(t) \le \left[c_1 + c_2\int_0^t f_2(s)\exp\left(\int_0^s f_3(\tau)\,d\tau\right)ds\right]$$

$$\times \exp\left\{\int_0^t f_1(s)\,ds + \int_0^t f_2(s)\exp\left(\int_0^s f_4(\tau)\,d\tau\right)\right.$$

$$\left.\times\left[\int_0^s f_3(\tau)\exp\left(\int_0^\tau f_1(\sigma)\,d\sigma\right)d\tau\right]ds\right\},\ t>0.$$

∎

Lemma 6.1.1 *Let* $u_1(t),\ u_2(t),\ f_1(t),\ f_2(t),\ f_3(t),\ f_4(t) \in C([0,1],\mathbb{R}_+)$, $t^{\gamma_1}f_1(t),\ t^{\gamma_2}f_2(t),\ t^{\gamma_1}f_3(t),\ t^{\gamma_2}f_4(t) \in C([1,\infty),\mathbb{R}_+)$ *and* $c_i, d_i, \gamma_i > 0,\ i=1,2$. *If*

$$u_1(t) \le c_1 + d_1 t^{\gamma_1} + t^{\gamma_1}\left[\int_0^t f_1(s)\,u_1(s)\,ds + \int_0^t f_2(s)\,u_2(s)\,ds\right],\ t>0, \quad (6.13)$$

and

$$u_2(t) \le c_2 + d_2 t^{\gamma_2} + t^{\gamma_2}\left[\int_0^t f_3(s)\,u_1(s)\,ds + \int_0^t f_4(s)\,u_2(s)\,ds\right],\ t>0, \quad (6.14)$$

then for $0<t<1$

$$u_1(t) \le \left[c_1 + d_1 + (c_2+d_2)\int_0^t f_2(s)\exp\left(\int_0^s f_3(\tau)\,d\tau\right)ds\right]$$

$$\times \exp\left\{\int_0^t f_1(s)\,ds + \int_0^t f_2(s)\exp\left(\int_0^s f_4(\tau)\,d\tau\right)\right.$$

$$\left. \times \left(\int_0^s f_3(\tau)\exp\left(\int_0^\tau f_1(\sigma)\,d\sigma\right)d\tau\right)ds\right\}, \tag{6.15}$$

$$u_2(t) \le \left[(c_2+d_2) + (c_1+d_1)\int_0^t f_3(s)\exp\left(\int_0^s f_1(\tau)\,d\tau\right)ds\right]$$

$$\times \exp\left\{\int_0^t f_4(s)\,ds + \int_0^t f_3(s)\exp\left(\int_0^s f_1(\tau)\,d\tau\right)\right.$$

$$\left. \times \left(\int_0^s f_2(\tau)\exp\left(\int_0^\tau f_4(\sigma)\,d\sigma\right)d\tau\right)ds\right\}, \tag{6.16}$$

and for $t \ge 1$

$$u_1(t) \le t^{\gamma_1}\left[b_1 + b_2\int_1^t s^{\gamma_2} f_2(s)\exp\left(\int_1^s \tau^{\gamma_1} f_3(\tau)\,d\tau\right)ds\right]$$

$$\times \exp\left\{\int_1^t s^{\gamma_1} f_1(s)\,ds + \int_1^t s^{\gamma_2} f_2(s)\exp\left(\int_1^s \tau^{\gamma_2} f_4(\tau)\,d\tau\right)\right.$$

$$\left. \times \left(\int_1^s \tau^{\gamma_1} f_3(\tau)\exp\left(\int_1^\tau \sigma^{\gamma_1} f_1(\sigma)\,d\sigma\right)d\tau\right)ds\right\}, \tag{6.17}$$

and

$$u_2(t) \le t^{\gamma_2}\left[b_2 + b_1\int_1^t s^{\gamma_1} f_3(s)\exp\left(\int_1^s \tau^{\gamma_1} f_1(\tau)\,d\tau\right)ds\right]$$

$$\times \exp\left\{\int_1^t s^{\gamma_2} f_4(s)\,ds + \int_1^t s^{\gamma_1} f_3(s)\exp\left(\int_1^s \tau^{\gamma_1} f_1(\tau)\,d\tau\right)\right.$$

$$\left. \times \left(\int_1^s \tau^{\gamma_2} f_2(\tau)\exp\left(\int_1^\tau \sigma^{\gamma_2} f_4(\sigma)\,d\sigma\right)d\tau\right)ds\right\}. \tag{6.18}$$

Here

$$b_1 = c_1 + d_1 + \int_0^1 f_1(s)\,ds\left[c_1 + d_1 + (c_2+d_2)\int_0^1 f_2(s)\exp\left(\int_0^1 f_3(\tau)\,d\tau\right)ds\right]$$

$$\times \exp \left\{ \int_0^1 f_1(s)\, ds + \int_0^1 f_2(s) \exp \left(\int_0^1 f_4(\tau)\, d\tau \right) \right.$$

$$\left. \times \left(\int_0^1 f_3(\tau) \exp \left(\int_0^1 f_1(\sigma)\, d\sigma \right) d\tau \right) ds \right\}$$

$$+ \int_0^1 f_2(s)\, ds \left[c_2 + d_2 + (c_1 + d_1) \int_0^1 f_3(s) \exp \left(\int_0^1 f_1(\tau)\, d\tau \right) ds \right]$$

$$\times \exp \left\{ \int_0^1 f_4(s)\, ds + \int_0^1 f_3(s) \exp \left(\int_0^1 f_1(\tau)\, d\tau \right) \right.$$

$$\left. \times \left(\int_0^1 f_2(\tau) \exp \left(\int_0^1 f_4(\sigma)\, d\sigma \right) d\tau \right) ds \right\},$$

and

$$b_2 = c_2 + d_2 + \int_0^1 f_3(s)\, ds \left[c_1 + d_1 + (c_2 + d_2) \int_0^1 f_2(s) \exp \left(\int_0^1 f_3(\tau)\, d\tau \right) ds \right]$$

$$\times \exp \left\{ \int_0^1 f_1(s)\, ds + \int_0^1 f_2(s) \exp \left(\int_0^1 f_4(\tau)\, d\tau \right) \right.$$

$$\left. \times \left(\int_0^1 f_3(\tau) \exp \left(\int_0^1 f_1(\sigma)\, d\sigma \right) d\tau \right) ds \right\}$$

$$+ \int_0^1 f_4(s)\, ds \left[c_2 + d_2 + (c_1 + d_1) \int_0^1 f_3(s) \exp \left(\int_0^1 f_1(\tau)\, d\tau \right) ds \right]$$

$$\times \exp \left\{ \int_0^1 f_4(s)\, ds + \int_0^1 f_3(s) \exp \left(\int_0^1 f_1(\tau)\, d\tau \right) \right.$$

$$\left. \times \left(\int_0^1 f_2(\tau) \exp \left(\int_0^1 f_4(\sigma)\, d\sigma \right) d\tau \right) ds \right\}.$$

Proof. For $0 < t < 1$, (6.13) and (6.14) imply that

$$u_1(t) \le c_1 + d_1 + \int_0^t f_1(s)\, u_1(s)\, ds + \int_0^t f_2(s)\, u_2(s)\, ds,$$

$$u_2(t) \le c_2 + d_2 + \int_0^t f_3(s)\, u_1(s)\, ds + \int_0^t f_4(s)\, u_2(s)\, ds,$$

and (6.15), (6.16) follow directly from Theorem 6.1.1. For $t \ge 1$ we find from (6.13) and (6.14) the estimate

$$\frac{u_1(t)}{t^{\gamma_1}} \le c_1 + d_1 + \int_0^t f_1(s)\, u_1(s)\, ds + \int_0^t f_2(s)\, u_2(s)\, ds,$$

$$\frac{u_2(t)}{t^{\gamma_2}} \le c_2 + d_2 + \int_0^t f_3(s)\, u_1(s)\, ds + \int_0^t f_4(s)\, u_2(s)\, ds,$$

or

$$\frac{u_1(t)}{t^{\gamma_1}} \le b_1 + \int_1^t s^{\gamma_1} f_1(s)\, \frac{u_1(s)}{s^{\gamma_1}} ds + \int_1^t s^{\gamma_2} f_2(s)\, \frac{u_2(s)}{s^{\gamma_2}} ds,$$

$$\frac{u_2(t)}{t^{\gamma_2}} \le b_2 + \int_1^t s^{\gamma_1} f_3(s)\, \frac{u_1(s)}{s^{\gamma_1}} ds + \int_1^t s^{\gamma_2} f_4(s)\, \frac{u_2(s)}{s^{\gamma_2}} ds.$$

Now (6.17) and (6.18) follow immediately from Theorem 6.1.1. ∎

6.2 Asymptotic behavior for a system involving Riemann-Liouville fractional derivatives

In this section, we will discuss the asymptotic behaviour of global solutions to the system (6.1) containing Riemann-Liouville fractional derivatives

$$\begin{cases} D_0^{1+\alpha_1} y_1(t) = g_1\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right), \ 0 \le \beta_1 < \alpha_1 < 1,\ t > 0, \\ D_0^{1+\alpha_2} y_2(t) = g_2\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right), \ 0 \le \beta_2 < \alpha_2 < 1,\ t > 0, \end{cases} \tag{6.19}$$

with initial conditions

$$\begin{cases} I_0^{1-\alpha_1} y_1(0) = b_1,\ D_0^{\alpha_1} y_1(0) = b_2,\ b_1, b_2 \in \mathbb{R}, \\ I_0^{1-\alpha_2} y_2(0) = c_1,\ D_0^{\alpha_2} y_2(0) = c_2,\ c_1, c_2 \in \mathbb{R}, \end{cases} \tag{6.20}$$

in the product spaces $C^{1+\alpha_1}_{1-\alpha_1}[0,\infty) \times C^{1+\alpha_2}_{1-\alpha_2}[0,\infty)$, where $C^{1+\alpha}_{1-\alpha}[0,\infty)$ is as in (3.32). In the sequel, we suppose that the functions g_i, $i = 1,2$, satisfy the following conditions

(AR1) The functions $g_i(., u_1(.), u_2(.)) : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2$, are such that $g_i(., u_1(.), u_2(.)) \in C_{1-\alpha_i}[0,\infty)$, for any $u_i \in C_{1-\alpha_i}[0,\infty)$, $i = 1,2$.

(AR2) There exist continuous functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,3,4$, such that

$$|g_1(t, u(t), v(t))| \leq f_1(t)|u(t)| + f_2(t)|v(t)|,\ t > 0, \tag{6.21}$$

$$|g_2(t, u(t), v(t))| \leq f_3(t)|u(t)| + f_4(t)|v(t)|,\ t > 0. \tag{6.22}$$

The next result provides useful estimates for solutions of Problem (6.19)-(6.20).

Lemma 6.2.1 *Assume that $(y_1, y_2) \in C_{1-\alpha_1}[0,\infty) \times C_{1-\alpha_2}[0,\infty)$ is a solution of Problem (6.19)-(6.20) and g_i, $i = 1,2$, satisfy **(AR1)-(AR2)**. Then, we have*

$$t^{1-(\alpha_1-\beta_1)} \left| D_0^{\beta_1} y_1(t) \right| \leq z_1(t), \quad t > 0, \tag{6.23}$$

$$t^{1-(\alpha_2-\beta_2)} \left| D_0^{\beta_2} y_2(t) \right| \leq z_2(t), \quad t > 0, \tag{6.24}$$

where

$$z_1(t) = B_1 + B_2 t + B_3 t \int_0^t \left[f_1(s) \left| D_0^{\beta_1} y_1(s) \right| + f_2(s) \left| D_0^{\beta_2} y_2(s) \right| \right] ds, \ t > 0, \tag{6.25}$$

$$z_2(t) = C_1 + C_2 t + C_3 t \int_0^t \left[f_3(s) \left| D_0^{\beta_1} y_1(s) \right| + f_4(s) \left| D_0^{\beta_2} y_2(s) \right| \right] ds, \ t > 0, \tag{6.26}$$

$$B_1 = \frac{|b_1|}{\Gamma(\alpha_1 - \beta_1)}, \ B_3 = \frac{1}{\Gamma(1 + \alpha_1 - \beta_1)}, \ B_2 = |b_2| B_3,$$

and

$$C_1 = \frac{|c_1|}{\Gamma(\alpha_2 - \beta_2)}, \ C_3 = \frac{1}{\Gamma(1 + \alpha_2 - \beta_2)}, \ C_2 = |c_2| C_3.$$

Proof. Applying I_0^1 to the first equation of System (6.19), we obtain

$$D_0^{\alpha_1} y_1(t) = b_2 + I_0^1 g_1 \left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t) \right), \ t > 0. \tag{6.27}$$

Since $g_1 \in C_{1-\alpha_1}[0, \infty)$, (6.19) implies that $D_0^{1+\alpha_1} y_1 = D^2 I_0^{1-\alpha_1} y_1 \in C_{1-\alpha_1}[0, \infty)$. Therefore $I_0^{1-\alpha_1} y_1 \in C_{1-\alpha_1}^2[0, \infty) \subset C_{1-\alpha_1}^1[0, \infty)$, see Lemma 3.1.1. In view of Lemma 3.1.2, we infer that

$$D_0^{\beta_1} y_1(t) = \frac{b_1}{\Gamma(\alpha_1 - \beta_1)} t^{\alpha_1 - \beta_1 - 1} + I_0^{\alpha_1 - \beta_1} D_0^{\alpha_1} y_1(t), \quad t > 0. \tag{6.28}$$

Let us insert the expression (6.27) into (6.28), use Property 2.2.1 and Lemma 2.2.2, we find

$$D_0^{\beta_1} y_1(t) = \frac{b_1}{\Gamma(\alpha_1 - \beta_1)} t^{\alpha_1 - \beta_1 - 1} + I_0^{\alpha_1 - \beta_1} \left[b_2 + I_0^1 g_1 \left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t) \right) \right]$$

$$= \frac{b_1}{\Gamma(\alpha_1-\beta_1)} t^{\alpha_1-\beta_1-1} + \frac{b_2}{\Gamma(1+\alpha_1-\beta_1)} t^{\alpha_1-\beta_1} + I_0^{1+\alpha_1-\beta_1} g_1\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right)$$

$$= \frac{b_1}{\Gamma(\alpha_1-\beta_1)} t^{\alpha_1-\beta_1-1} + \frac{b_2}{\Gamma(1+\alpha_1-\beta_1)} t^{\alpha_1-\beta_1}$$

$$+ \frac{1}{\Gamma(1+\alpha_1-\beta_1)} \int_0^t (t-s)^{\alpha_1-\beta_1} g_1\left(s, D_0^{\beta_1} y_1(s), D_0^{\beta_2} y_2(s)\right) ds, \quad t>0. \tag{6.29}$$

Multiplying both sides of (6.29) by $t^{1-(\alpha_1-\beta_1)}$ and using (6.21) we deduce

$$t^{1-(\alpha_1-\beta_1)} \left| D_0^{\beta_1} y_1(t) \right| \leq B_1 + B_2 t$$

$$+ B_3 t \int_0^t \left[f_1(s) \left| D_0^{\beta_1} y_1(s) \right| + f_2(s) \left| D_0^{\beta_2} y_2(s) \right| \right] ds, \quad t>0. \tag{6.30}$$

Similarly, we can prove that

$$t^{1-(\alpha_2-\beta_2)} \left| D_0^{\beta_2} y_2(t) \right| \leq C_1 + C_2 t$$

$$+ C_3 t \int_0^t \left[f_3(s) \left| D_0^{\beta_1} y_1(s) \right| + f_4(s) \left| D_0^{\beta_2} y_2(s) \right| \right] ds, \quad t>0. \tag{6.31}$$

Thus (6.23) and (6.24) follow directly from (6.25), (6.26), (6.30) and (6.31). ∎

Lemma 6.2.2 *Let $(y_1, y_2) \in C_{1-\alpha_1}[0,\infty) \times C_{1-\alpha_2}[0,\infty)$ be a solution of Problem (6.19)-(6.20) and g_i, $i=1,2$, satisfy **(AR1)** and **(AR2)** with*

$$t^{(\alpha_1-\beta_1)-1} f_1(t), \; t^{(\alpha_2-\beta_2)-1} f_2(t) \in L(0,1), \tag{6.32}$$

$$t^{(\alpha_1-\beta_1)-1} f_3(t), \; t^{(\alpha_2-\beta_2)-1} f_4(t) \in L(0,1), \tag{6.33}$$

$$t^{(\alpha_1-\beta_1)}f_1(t),\ t^{(\alpha_2-\beta_2)}f_2(t)\in L_1(1,\infty), \tag{6.34}$$

and

$$t^{(\alpha_1-\beta_1)}f_3(t),\ t^{(\alpha_2-\beta_2)}f_4(t)\in L_1(1,\infty). \tag{6.35}$$

Then

$$\lim_{t\to\infty}\int_0^t g_i\left(s, D_0^{\beta_1}y_1(s), D_0^{\beta_2}y_2(s)\right)<\infty,\ i=1,2.$$

Proof. Inserting the terms $s^{1-(\alpha_1-\beta_1)}s^{(\alpha_1-\beta_1)-1}$ and $s^{1-(\alpha_2-\beta_2)}s^{(\alpha_2-\beta_2)-1}$ inside the integrals of (6.25), (6.26), respectively, and using Lemma 6.2.1, give

$$z_1(t)\le B_1+B_2t+B_3t\int_0^t\left[s^{(\alpha_1-\beta_1)-1}f_1(s)z_1(s)+s^{(\alpha_2-\beta_2)-1}f_2(s)z_2(s)\right]ds,\ t>0,$$

and

$$z_2(t)\le C_1+C_2t+C_3t\int_0^t\left[s^{(\alpha_1-\beta_1)-1}f_3(s)z_1(s)+s^{(\alpha_2-\beta_2)-1}f_4(s)z_2(s)\right]ds,\ t>0.$$

By virtue of Lemma 6.1.1, (6.32), (6.33), (6.34) and (6.35) we get

$$z_i(t)\le\begin{cases}K_i,\ 0\le t<1,\ i=1,2\\ A_it,\ t\ge 1,\ i=1,2,\end{cases}$$

for some positive constants K_i and A_i, $i=1,2$. Therefore in view of Lemma 6.2.1, we find

$$\left|D_0^{\beta_1}y_1(t)\right|\le\begin{cases}K_1t^{(\alpha_1-\beta_1)-1},\ 0<t<1,\\ A_1t^{(\alpha_1-\beta_1)},\ t\ge 1,\end{cases}$$

and

$$\left|D_0^{\beta_2} y_2(t)\right| \le \begin{cases} K_2 t^{(\alpha_2-\beta_2)-1}, & 0<t<1, \\ A_2 t^{(\alpha_2-\beta_2)}, & t \ge 1. \end{cases}$$

On the other hand, by our assumption (6.21) we see that

$$\int_0^t \left|g_1\left(s, D_0^{\beta_1} y_1(s), D_0^{\beta_2} y_2(s)\right)\right| ds \le \int_0^t \left(f_1(s)\left|D_0^{\beta_1} y_1(s)\right| + f_2(s)\left|D_0^{\beta_2} y_2(s)\right|\right) ds$$

$$\le \int_0^1 \left(f_1(s)\left|D_0^{\beta_1} y_1(s)\right| + f_2(s)\left|D_0^{\beta_2} y_2(s)\right|\right) ds$$

$$+\int_1^t \left(f_1(s)\left|D_0^{\beta_1} y_1(s)\right| + f_2(s)\left|D_0^{\beta_2} y_2(s)\right|\right) ds$$

$$\le K_1 \int_0^1 s^{(\alpha_1-\beta_1)-1} f_1(s)\, ds + K_2 \int_0^1 t^{(\alpha_2-\beta_2)-1} f_2(s)\, ds$$

$$+A_1 \int_1^t s^{(\alpha_1-\beta_1)} f_1(s)\, ds + A_2 \int_1^t s^{(\alpha_2-\beta_2)} f_2(s)\, ds, \quad t>0.$$

Again by (6.32) and (6.34) we deduce

$$\int_0^t \left|g_1\left(s, D_0^{\beta_1} y_1(s), D_0^{\beta_2} y_2(s)\right)\right| ds < \infty.$$

By the same way we can prove

$$\int_0^t \left|g_2\left(s, D_0^{\beta_1} y_1(s), D_0^{\beta_2} y_2(s)\right)\right| ds < \infty,$$

and the result follows. ∎

The main result is given by the following theorem.

Theorem 6.2.1 *Under the same hypotheses as in Lemma 6.2.2, any solution*

$(y_1, y_2) \in C_{1-\alpha_1}[0,\infty) \times C_{1-\alpha_2}[0,\infty)$ *of Problem (6.19)-(6.20) has the following property*

$$\lim_{t\to\infty} \frac{y_i(t)}{t^{\alpha_i}} = a_i, \ \textit{for some } a_i \in \mathbb{R}, \ i = 1, 2.$$

Proof. It is clear, by virtue of (6.27) and Lemma 6.2.2, that there exists $r_1 \in \mathbb{R}$, such that

$$\lim_{t\to\infty} D_0^{\alpha_1} y_1(t) = r_1.$$

Finally, by Lemma 3.1.5, we conclude that

$$\lim_{t\to\infty} \frac{y_1(t)}{t^{\alpha_1}} = \lim_{t\to\infty} \frac{D_0^{\alpha_1} y_1(t)}{\Gamma(1+\alpha_1)} = \frac{r_1}{\Gamma(1+\alpha_1)} = a_1 \in \mathbb{R}.$$

The second part is proved similarly. ∎

6.3 Asymptotic behavior for a system involving fractional derivatives of Caputo type

In this section, we will discuss the asymptotic behaviour of global solutions to the system of equations containing Caputo fractional derivatives

$$\begin{cases} \left({}^C D_0^{\alpha_1} y_1\right)'(t) = g_1\left(t, {}^C D_0^{\beta_1} y_1(t), {}^C D_0^{\beta_2} y_2(t)\right), & 0 \le \beta_1 < \alpha_1 < 1, \, t > 0, \\ \left({}^C D_0^{\alpha_2} y_2\right)'(t) = g_2\left(t, {}^C D_0^{\beta_1} y_1(t), {}^C D_0^{\beta_2} y_2(t)\right), & 0 \le \beta_2 < \alpha_2 < 1, \, t > 0, \end{cases} \tag{6.36}$$

with initial conditions

$$\begin{cases} {}^{C}D_{0}^{\alpha_1} y_1(t)\,|_{t=0} = b_1 \text{ and } y_1(t)\,|_{t=0} = b_2,\ \ b_1, b_2 \in \mathbb{R}, \\ {}^{C}D_{0}^{\alpha_2} y_2(t)\,|_{t=0} = c_1 \text{ and } y_2(t)\,|_{t=0} = c_2,\ \ c_1, c_2 \in \mathbb{R}, \end{cases} \tag{6.37}$$

in product spaces $C_{1-\alpha_1}^{\alpha_1,1}[0,\infty) \times C_{1-\alpha_1}^{\alpha_1,1}[0,\infty)$, where $C_{1-\alpha}^{\alpha,1}[0,\infty)$ defined in (3.62). We suppose that functions g_i, $i = 1, 2$, satisfy the following conditions:

(AC1) The functions $g_i : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are such that $g_i(., u_1(.), u_2(.)) \in C_{1-\alpha_i}[0,\infty)$, $i = 1, 2$, for any $u_i \in AC[0,\infty)$, $i = 1, 2$.

(AC2) There exist continuous functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3, 4$, such that

$$|g_1(t, u_1(t), u_2(t))| \le f_1(t)\,|u_1(t)| + f_2(t)\,|u_2(t)|\,,\ \ t > 0, \tag{6.38}$$

$$|g_2(t, u_1(t), u_2(t))| \le f_3(t)\,|u_1(t)| + f_4(t)\,|u_2(t)|\,,\ \ t > 0. \tag{6.39}$$

The next result provides useful estimates for solutions of Problem (6.36)-(6.37).

Lemma 6.3.1 *Let $(y_1, y_2) \in AC[0,\infty) \times AC[0,\infty)$ be a solution of Problem (6.36)-(6.37) and g_i, $i = 1, 2$, satisfy (**AC1**)-(**AC2**). Then*

$$\left|{}^{C}D_{0}^{\beta_i} y_i(t)\right| \le z_i(t)\,,\quad i = 1, 2,\quad t > 0, \tag{6.40}$$

where

$$z_1(t) = t^{\alpha_1-\beta_1}\left(B_1 + B_2\int_0^t \left[f_1(s)\left|{}^CD_0^{\beta_1}y_1(s)\right| + f_2(s)\left|{}^CD_0^{\beta_2}y_2(s)\right|\right]ds\right),\ t>0, \tag{6.41}$$

$$z_2(t) = t^{\alpha_2-\beta_2}\left(C_1 + C_2\int_0^t \left[f_3(s)\left|{}^CD_0^{\beta_1}y_1(s)\right| + f_4(s)\left|{}^CD_0^{\beta_2}y_2(s)\right|\right]ds\right),\ t>0, \tag{6.42}$$

$$B_1 = |b_1|\,B_2,\ B_2 = \frac{1}{\Gamma(\alpha_1-\beta_1+1)},$$

and

$$C_1 = |c_1|\,C_2,\ C_2 = \frac{1}{\Gamma(\alpha_2-\beta_2+1)}.$$

Proof. Applying I_0^1 to the first equation of System (6.36), we obtain

$$\begin{aligned}{}^CD_0^{\alpha_1}y_1 &= b_1 + I_0^1 g_1\left(t, {}^CD_0^{\beta_1}y_1(t), {}^CD_0^{\beta_2}y_2(t)\right)\\ &= b_1 + \int_0^t g_1\left(s, {}^CD_0^{\beta_1}y_1(s), {}^CD_0^{\beta_2}y_2(s)\right)ds,\ t>0.\end{aligned} \tag{6.43}$$

In virtue of Lemma 3.1.3 and (6.43), we find

$$\begin{aligned}{}^CD_0^{\beta_1}y_1(t) &= I_0^{\alpha_1-\beta_1}\left({}^CD_0^{\alpha_1}y_1(t)\right)\\ &= I_0^{\alpha_1-\beta_1}\left[b_1 + I_0^1 g_1\left(s, {}^CD_0^{\beta_1}y_1(s), {}^CD_0^{\beta_2}y_2(s)\right)\right](t),\quad t>0.\end{aligned}$$

Applying Property 2.2.1 and Lemma 2.2.2, lead to

$$
{}^{C}D_{0}^{\beta_1}y_1(t) = \frac{b_1}{\Gamma(\alpha_1-\beta_1+1)}t^{\alpha_1-\beta_1} + I_{0}^{\alpha_1-\beta_1+1}g_1\left(t, {}^{C}D_{0}^{\beta_1}y_1(t), {}^{C}D_{0}^{\beta_2}y_2(t)\right)
$$

$$
= B_2\left(b_1 t^{\alpha_1-\beta_1} + \int_0^t (t-s)^{\alpha_1-\beta_1} g_1\left(s, {}^{C}D_{0}^{\beta_1}y_1(s), {}^{C}D_{0}^{\beta_2}y_2(s)\right) ds\right), \quad t>0.
$$

Using (6.38) we deduce

$$
\left|{}^{C}D_{0}^{\beta_1}y_1(t)\right| \leq B_1 t^{\alpha_1-\beta_1}
$$

$$
+B_2 t^{\alpha_1-\beta_1}\int_0^t \left[f_1(s)\left|{}^{C}D_{0}^{\beta_1}y_1(s)\right| + f_2(s)\left|{}^{C}D_{0}^{\beta_2}y_2(s)\right|\right] ds,\ t>0. \tag{6.44}
$$

Similarly, we see that

$$
\left|{}^{C}D_{0}^{\beta_2}y_2(t)\right| \leq C_1 t^{\alpha_2-\beta_2}
$$

$$
+C_2 t^{\alpha_2-\beta_2}\int_0^t \left[f_3(s)\left|{}^{C}D_{0}^{\beta_1}y_1(s)\right| + f_4(s)\left|{}^{C}D_{0}^{\beta_2}y_2(s)\right|\right] ds,\ t>0. \tag{6.45}
$$

Now the result follows directly from (6.41), (6.42), (6.44) and (6.45). ∎

Lemma 6.3.2 *Under the same hypotheses as in Lemma 6.3.1 with*

$$
t^{\alpha_1-\beta_1}f_1(t),\ t^{\ \alpha_2-\beta_2}f_2(t),\ t^{\alpha_1-\beta_1}f_3(t),\ t^{\alpha_2-\beta_2}f_4(t) \in L_1(0,\infty), \tag{6.46}
$$

we have

$$
\lim_{t\to\infty}\int_0^t g_i\left(s, {}^{C}D_{0}^{\beta_1}y_1(s), {}^{C}D_{0}^{\beta_2}y_2(s)\right) ds < \infty,\ i=1,2.
$$

Proof. In view of Lemma 6.3.1, we deduce

$$z_1(t) \le t^{\alpha_1-\beta_1}\left(B_1 + B_2\int_0^t [f_1(s)\,z_1(s) + f_2(s)\,z_2(s)]\,ds\right), \quad t>0, \tag{6.47}$$

and

$$z_2(t) \le t^{\alpha_2-\beta_2}\left(C_1 + C_2\int_0^t [f_3(s)\,z_1(s) + f_4(s)\,z_2(s)]\,ds\right), \ t>0. \tag{6.48}$$

Inserting the terms $s^{\alpha_1-\beta_1}s^{-(\alpha_1-\beta_1)}$ and $s^{\alpha_2-\beta_2}s^{-(\alpha_2-\beta_2)}$ inside the integrals of (6.47) and (6.48), respectively, give

$$\frac{z_1(t)}{t^{\alpha_1-\beta_1}} \le B_1 + B_2\int_0^t \left(s^{\alpha_1-\beta_1}f_1(s)\,\frac{z_1(s)}{s^{\alpha_1-\beta_1}} + s^{\alpha_2-\beta_2}f_2(s)\,\frac{z_2(s)}{s^{\alpha_2-\beta_2}}\right)ds,\ t>0,$$

and

$$\frac{z_2(t)}{t^{\alpha_2-\beta_2}} \le C_1 + C_2\int_0^t \left(s^{\alpha_1-\beta_1}f_3(s)\,\frac{z_1(s)}{s^{\alpha_1-\beta_1}} + s^{\alpha_2-\beta_2}f_4(s)\,\frac{z_2(s)}{s^{\alpha_2-\beta_2}}\right)ds,\ t>0.$$

By virtue of Theorem 6.1.1 and (6.46) we get

$$z_i(t) \le A_i t^{\alpha_i-\beta_i},\ i=1,2,\ t>0,$$

for some positive constants A_i, $i=1,2$. Again by Lemma 6.3.1 we obtain

$$\left|{}^{C}D_0^{\beta_i}y_i(t)\right| \le A_i t^{\alpha_i-\beta_i},\ i=1,2,\ t>0.$$

On the other hand, by our assumption (6.38) and (6.46) we see that

$$\int_0^t \left|g_1\left(s, {}^C D_0^{\beta_1} y_1(s), {}^C D_0^{\beta_2} y_2(s)\right)\right| ds$$

$$\leq \int_0^t \left[f_1(s)\left|{}^C D_0^{\beta_1} y_1(s)\right| + f_2(s)\left|{}^C D_0^{\beta_2} y_2(s)\right|\right] ds$$

$$\leq A_1 \int_0^t s^{\alpha_1-\beta_1} f_1(s)\, ds + A_2 \int_0^t s^{\alpha_2-\beta_2} f_2(s)\, ds < \infty.$$

Similarly

$$\int_0^t \left|g_2\left(s, {}^C D_0^{\beta_1} y_1(s), {}^C D_0^{\beta_2} y_2(s)\right)\right| ds$$

$$\leq \int_0^t \left[f_3(s)\left|{}^C D_0^{\beta_1} y_1(s)\right| + f_4(s)\left|{}^C D_0^{\beta_2} y_2(s)\right|\right] ds$$

$$\leq A_1 \int_0^t s^{\alpha_1-\beta_1} f_3(s)\, ds + A_2 \int_0^t s^{\alpha_2-\beta_2} f_4(s)\, ds < \infty,$$

and the result follows. ∎

Theorem 6.3.1 *Under the same hypotheses as in Lemmas 6.3.1 and 6.3.2, any solution* $(y_1, y_2) \in AC[0,\infty) \times AC[0,\infty)$ *of Problem (6.36)-(6.37) has the following property*

$$\lim_{t\to\infty} \frac{y_i(t)}{t^{\alpha_i}} = a_i \in \mathbb{R},\ \ i = 1, 2.$$

Proof. In virtue of (6.43) and Lemma 6.3.2, we deduce that there exist $r_1 \in \mathbb{R}$ such that

$$\lim_{t\to\infty} {}^C D_0^{\alpha_1} y_1(t) = r_1.$$

Further, by Lemma 3.1.6, we conclude that

$$\lim_{t\to\infty}\frac{y_1(t)}{t^{\alpha_1}}=\lim_{t\to\infty}\frac{{}^{C}D_0^{\alpha_1}y_1(t)}{\Gamma(1+\alpha_1)}=\frac{r_1}{\Gamma(1+\alpha_1)}=a_1\in\mathbb{R}.$$

In the same way we have

$$\lim_{t\to\infty}\frac{y_2(t)}{t^{\alpha_2}}=a_2\in\mathbb{R}.$$

∎

CHAPTER 7

POWER TYPE DECAY AND BOUNDEDNESS FOR A SYSTEM

In this chapter, we consider a class of systems that involve Riemann-Liouville and Caputo derivatives. We determine sufficient conditions on the right hand sides of the systems guaranteeing global existence.

7.1 Power type decay in case of Riemann-Liouville fractional derivatives

In this section, we consider the system

$$\begin{cases} D_0^{\alpha_1} y_1(t) = g_1\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right) \\ D_0^{\alpha_2} y_2(t) = g_2\left(t, D_0^{\beta_1} y_1(t), D_0^{\beta_2} y_2(t)\right) \end{cases}, \quad 0 \leq \beta_1, \beta_2 < \min\{\alpha_1, \alpha_2\} < 1,\ t > 0, \tag{7.1}$$

subject to the initial conditions

$$I_0^{1-\alpha_i} y_i(t)\,|_{t=0} = b_i,\ b_i \in \mathbb{R},\ i = 1, 2, \tag{7.2}$$

in the space $C_{1-\alpha_1}^{\alpha_1}[0,\infty) \times C_{1-\alpha_2}^{\alpha_2}[0,\infty)$, where $C_{1-\alpha}^{\alpha}[0,\infty)$ is as in (4.16). The functions $g_i : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are continuous functions, such that

(SSR1) $g_i(., u_1(.), u_2(.)) \in C_{1-\alpha_i}[0,\infty)$, $i = 1, 2$, for any $u_i \in C_{1-\alpha_i}[0,\infty)$, $i = 1, 2$.

(SSR2) There exist continuous functions $f_i : [0,\infty) \to [0,\infty)$, $i = 1, 2, 3, 4$, such that

$$|g_1(t, u_1(t), u_2(t))| \leq e^{-\gamma_1 t}[f_1(t)|u_1(t)| + f_2(t)|u_2(t)|], \ t > 0, \tag{7.3}$$

$$|g_2(t, u_1(t), u_2(t))| \leq e^{-\gamma_2 t}[f_3(t)|u_1(t)| + f_4(t)|u_2(t)|], \ t > 0, \tag{7.4}$$

for some $\gamma_i \in \mathbb{R}$, $i = 1, 2$.

Lemma 7.1.1 *Assume that* $(y_1, y_2) \in C_{1-\alpha_1}[0, \infty) \times C_{1-\alpha_2}[0, \infty)$ *is a solution of Problem (7.1)-(7.2) and* g_i, $i = 1, 2$, *satisfy* ***(SSR1)-(SSR2)***. *Then*

$$t^{1-(\alpha_i-\beta_i)} \left|D_0^{\beta_i} y_i(t)\right| \leq z_i(t), \ i = 1, 2, \ t > 0, \tag{7.5}$$

where

$$z_1(t) = K_1 \Bigg\{ |b_1| + t^{1-(\alpha_1-\beta_1)} \times \int_0^t (t-s)^{\alpha_1-\beta_1-1} e^{-\gamma_1 s} \left[f_1(s)\left|D_0^{\beta_1} y_1(s)\right| + f_2(s)\left|D_0^{\beta_2} y_2(s)\right|\right] ds \Bigg\}, \ t > 0, \tag{7.6}$$

$$z_2(t) = K_2 \Bigg\{ |b_2| + t^{1-(\alpha_2-\beta_2)} \times \int_0^t (t-s)^{\alpha_2-\beta_2-1} e^{-\gamma_2 s} \left[f_3(s)\left|D_0^{\beta_1} y_1(s)\right| + f_4(s)\left|D_0^{\beta_2} y_2(s)\right|\right] ds \Bigg\}, t > 0, \tag{7.7}$$

and

$$K_i = \frac{1}{\Gamma(\alpha_i - \beta_i)}, \ i = 1, 2.$$

Proof. Because $g_1 \in C_{1-\alpha_1}[0,\infty)$, (7.1) implies that $D_0^{\alpha_1} y_1 = D I_0^{1-\alpha_1} y_1 \in C_{1-\alpha_1}[0,\infty)$. Thus, by Lemma 3.1.1, $I_0^{1-\alpha_1} y_1 \in C_{1-\alpha_1}^1[0,\infty)$. By virtue of Lemma 3.1.2, we have

$$
\begin{aligned}
D_0^{\beta_1} y_1(t) &= \frac{I_0^{1-\alpha_1} y_1(0)}{\Gamma(\alpha_1-\beta_1)} t^{\alpha_1-\beta_1-1} + I_0^{\alpha_1-\beta_1} D_0^{\alpha_1} y_1(t) \\
&= \frac{b_1 t^{\alpha_1-\beta_1-1}}{\Gamma(\alpha_1-\beta_1)} + \frac{1}{\Gamma(\alpha_1-\beta_1)} \int_0^t (t-s)^{\alpha_1-\beta_1-1} D_0^{\alpha_1} y_1(s)\, ds \\
&= \frac{1}{\Gamma(\alpha_1-\beta_1)} \left[b_1 t^{\alpha_1-\beta_1-1} + \int_0^t (t-s)^{\alpha_1-\beta_1-1} g_1\left(s, D_0^{\beta_1} y_1(s), D_0^{\beta_2} y_2(s)\right) ds \right], \ t > 0,
\end{aligned} \tag{7.8}
$$

where b_1 comes from the initial condition in (7.2). Multiplying both sides of (7.8) by $t^{1-(\alpha_1-\beta_1)}$ and using the assumption (7.3), we obtain

$$
t^{1-(\alpha_1-\beta_1)} \left| D_0^{\beta_1} y_1(t) \right| \le z_1(t), \quad t > 0. \tag{7.9}
$$

Similarly, we see that

$$
t^{1-(\alpha_2-\beta_2)} \left| D_0^{\beta_2} y_2(t) \right| \le z_2(t), \quad t > 0. \tag{7.10}
$$

∎

Theorem 7.1.1 *Under the same hypotheses as in Lemma 7.1.1 with* $f_j \in L_q(0,\infty)$, $j = 1,2,3,4$, *for some* $q > \max\{\frac{1}{\alpha_1-\beta_1}, \frac{1}{\alpha_2-\beta_2}\}$ *and* $\gamma_i > 0$, $i = 1,2$**,** *there exist two positive constants* C_i, $i = 1,2$, *such that the solution of Problem*

(7.1)-(7.2) satisfies

$$\left| D_0^{\beta_i} y_i\left(t\right)\right| \leq C_i t^{\alpha_i-\beta_i-1}, \quad i=1,2, \quad t>0.$$

Proof. Inserting the terms $s^{\alpha_1-\beta_1-1}s^{1-(\alpha_1-\beta_1)}$ and $s^{\alpha_2-\beta_2-1}s^{1-(\alpha_2-\beta_2)}$ inside the integral in (7.6) and using (7.5) gives

$$z_1\left(t\right) \leq K_1\left|b_1\right| + K_1 t^{1-(\alpha_1-\beta_1)}$$

$$\times \int_0^t \left(t-s\right)^{\alpha_1-\beta_1-1} e^{-\gamma_1 s}\left[s^{\alpha_1-\beta_1-1} f_1\left(s\right) z_1\left(s\right) + s^{\alpha_2-\beta_2-1} f_2\left(s\right) z_2\left(s\right)\right] ds, \ t>0. \tag{7.11}$$

Applying Lemma 4.1.3 to (7.11), we infer that

$$z_1\left(t\right) \leq K_1\left|b_1\right| + K_1\left[r_1\left(\int_0^t \left[f_1\left(s\right) z_1\left(s\right)\right]^q ds\right)^{\frac{1}{q}} + r_2\left(\int_0^t \left[f_2\left(s\right) z_2\left(s\right)\right]^q ds\right)^{\frac{1}{q}}\right], \ t>0, \tag{7.12}$$

where

$$r_1 = \left[\max\left\{1, 2^{1-\lambda_1}\right\} \Gamma\left(\lambda_1\right)\left(2+\lambda_1\right)\left(p\gamma_1\right)^{-\lambda_1}\right]^{1/p},$$

$$r_2 = \left[\max\left\{1, 2^{1-\lambda_1}\right\} \Gamma\left(\lambda_2\right)\left(1+\frac{\lambda_2\left(\lambda_2+1\right)}{\lambda_1}\right)\left(p\gamma_1\right)^{-\lambda_2}\right]^{1/p},$$

$$\lambda_1 = p\left(\alpha_1-\beta_1-1\right)+1, \ \lambda_2 = p\left(\alpha_2-\beta_2-1\right)+1 \text{ and } 1/p+1/q=1.$$

Similarly, we have

$$z_2(t) \leq K_2|b_2| + K_2\left[r_3\left(\int_0^t [f_3(s)\,z_1(s)]^q\,ds\right)^{\frac{1}{q}} + r_4\left(\int_0^t [f_4(s)\,z_2(s)]^q\,ds\right)^{\frac{1}{q}}\right],\ t>0, \tag{7.13}$$

with

$$r_3 = \left[\max\left\{1, 2^{1-\lambda_2}\right\}\Gamma(\lambda_1)\left(1+\frac{\lambda_1(\lambda_1+1)}{\lambda_2}\right)(p\gamma_2)^{-\lambda_1}\right]^{1/p},$$

$$r_4 = \left[\max\left\{1, 2^{1-\lambda_2}\right\}\Gamma(\lambda_2)(2+\lambda_2)(p\gamma_2)^{-\lambda_2}\right]^{1/p},$$

$$\lambda_1 = p(\alpha_1-\beta_1-1)+1,\ \lambda_2 = p(\alpha_2-\beta_2-1)+1 \text{ and } 1/p+1/q=1.$$

Raising both sides of (7.12) and (7.13) to the power q and using Lemma 2.4.4, we get

$$z_1^q(t) \leq c_1 + \int_0^t [h_1(s)\,z_1^q(s) + h_2(s)\,z_2^q(s)]\,ds,\ t>0, \tag{7.14}$$

$$z_2^q(t) \leq c_2 + \int_0^t [h_3(s)\,z_1^q(s) + h_4(s)\,z_2^q(s)]\,ds,\ t>0, \tag{7.15}$$

where

$$c_i = 2^{q-1}K_i^q\,|b_i|^q,\ i=1,2,$$

$$h_i(t) = 2^{2(q-1)}[K_1 r_i f_i(t)]^q,\ i=1,2,\ t>0,$$

$$h_j(t) = 2^{2(q-1)}[K_2 r_j f_j(t)]^q,\ j=3,4,\ t>0.$$

Since $f_i \in L_q(0,\infty),\ i=1,2,3,4,$ we can apply Theorem 6.1.1 to (7.14) and (7.15), to get

$$z_i^q(t) \leq m_i,\ i=1,2,\ t>0,$$

for some positive constants m_i, $i = 1, 2$. In view of Lemma 7.1.1, we conclude that

$$\left| D_0^{\beta_i} y_i(t) \right| \leq C_i t^{\alpha_i - \beta_i - 1}, \ i = 1, 2, \ t > 0.$$

with $C_i = m_i^{1/q}$, $i = 1, 2$. In particular, when $\beta_1 = \beta_2 = 0$, we find

$$|y_i(t)| \leq C_i t^{\alpha_i - 1}, \ i = 1, 2, \ t > 0.$$

∎

7.2 Boundedness in case of Caputo fractional derivatives

In this section, we consider the following system

$$\begin{cases} {}^C D_0^{\alpha_1} y_1(t) = g_1\left(t, {}^C D_0^{\beta_1} y_1(t), {}^C D_0^{\beta_2} y_2(t)\right), \\ {}^C D_0^{\alpha_2} y_2(t) = g_2\left(t, {}^C D_0^{\beta_1} y_1(t), {}^C D_0^{\beta_2} y_2(t)\right), \end{cases} \quad 0 < \beta_1, \beta_2 < \min\{\alpha_1, \alpha_2\} < 1, \ t > 0, \tag{7.16}$$

subject to the initial conditions

$$y_i(t)\,|_{t=0} = b_i, \ b_i \in \mathbb{R}, \ i = 1, 2, \tag{7.17}$$

in the space $C^{\alpha_1}[0, \infty) \times C^{\alpha_2}[0, \infty)$, where $C^{\alpha}[0, \infty)$ is as in (4.33).

We consider the following assumptions on the functions g_i, $i = 1, 2$,

(SSC1) $g_i : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$, $i=1,2$, are such that $g_i\left(.,u_1\left(.\right),u_2\left(.\right)\right)\in C[0,\infty)$, $i=1,2$, for any $u_i\in C[0,\infty)$, $i=1,2$.

(SSC2) There exist continuous functions $f_i : [0,\infty)\to[0,\infty)$, $i=1,2,3,4$, such that

$$\left|g_1\left(t,u_1\left(t\right),u_2\left(t\right)\right)\right| \leq t^{\gamma_1}\left[f_1\left(t\right)\left|u_1\left(t\right)\right| + f_2\left(t\right)\left|u_2\left(t\right)\right|\right],\ \ t>0, \tag{7.18}$$

$$\left|g_2\left(t,u_1\left(t\right),u_2\left(t\right)\right)\right| \leq t^{\gamma_2}\left[f_3\left(t\right)\left|u_1\left(t\right)\right| + f_4\left(t\right)\left|u_2\left(t\right)\right|\right],\ \ t>0, \tag{7.19}$$

where $f_i\in L_q\left(0,\infty\right)$, $i=1,2,3,4$, for some $q>\max\left\{\frac{1}{\alpha_1-\beta_1},\frac{1}{\alpha_2-\beta_2}\right\}$, $\gamma_i=\frac{1}{q}-\left(\alpha_i-\beta_i\right)$, $i=1,2$.

The next results provide useful estimates for solutions of Problem (7.16)-(7.17).

Lemma 7.2.1 *Assume that* $(y_1,y_2)\in AC[0,\infty)\times AC[0,\infty)$ *is a solution of Problem (7.16)-(7.17) and* g_i*,* $i=1,2$*, satisfy* ***(SSC1)-(SSC2).*** *Then, we have*

$$\left|{}^{C}D_0^{\beta_i}y_i\left(t\right)\right| < z_i\left(t\right),\ \ i=1,2,\ \ t>0, \tag{7.20}$$

where

$$z_1\left(t\right)=K_1\left(\int_0^t\left[f_1^q\left(s\right)\left|{}^{C}D_0^{\beta_1}y_1\left(s\right)\right|^q + f_2^q\left(s\right)\left|{}^{C}D_0^{\beta_2}y_2\left(s\right)\right|^q\right]ds\right)^{\frac{1}{q}},\ \ t>0, \tag{7.21}$$

$$z_2\left(t\right)=K_2\left(\int_0^t\left[f_3^q\left(s\right)\left|{}^{C}D_0^{\beta_1}y_1\left(s\right)\right|^q + f_4^q\left(s\right)\left|{}^{C}D_0^{\beta_2}y_2\left(s\right)\right|^q\right]ds\right)^{\frac{1}{q}},\ \ t>0, \tag{7.22}$$

and

$$K_i = 2^{1/p} \frac{K^{1/p}_{p(\alpha_i - \beta_i - 1)+1, p\gamma_i}}{\Gamma(\alpha_i - \beta_i)}, \; K_{r,s} = \frac{\Gamma(r)\Gamma(s+1)}{\Gamma(r+s+1)}, \; p+q = pq, \; i = 1, 2.$$

Proof. In virtue of Lemma 3.1.3, we obtain

$${}^{C}D_{0}^{\beta_1} y_1(t) = I_0^{\alpha_1 - \beta_1} \; {}^{C}D_0^{\alpha_1} y_1(t) = \frac{1}{\Gamma(\alpha_1 - \beta_1)} \int_0^t (t-s)^{\alpha_1 - \beta_1 - 1} \; {}^{C}D_0^{\alpha_1} y_1(s)\, ds$$

$$= \frac{1}{\Gamma(\alpha_1 - \beta_1)} \int_0^t (t-s)^{\alpha_1 - \beta_1 - 1} g_1\left(s, {}^{C}D_0^{\beta 1} y_1(s), {}^{C}D_0^{\beta_2} y_2(s)\right) ds, \; t > 0. \quad (7.23)$$

Using the assumption (7.18), we find

$$\left| {}^{C}D_0^{\beta_1} y_1(t) \right|$$

$$\leq \frac{1}{\Gamma(\alpha_1 - \beta_1)} \int_0^t (t-s)^{\alpha_1 - \beta_1 - 1} s^{\gamma_1} \left[f_1(s) \left| {}^{C}D_0^{\beta_1} y_1(s) \right| + f_2(s) \left| {}^{C}D_0^{\beta_2} y_2(s) \right| \right] ds, \; t > 0.$$

Now, we can apply Lemma 4.1.2, because $\alpha_1 - \beta_1 > \frac{1}{q}$ and $1 + \gamma_1 = 1 + \frac{1}{q} - (\alpha_1 - \beta_1) > \frac{1}{q}$, to get

$$\left| {}^{C}D_0^{\beta_1} y_1(t) \right|$$

$$\leq \frac{K^{1/p}_{p(\alpha_1 - \beta_1 - 1)+1, p\gamma_1}}{\Gamma(\alpha_1 - \beta_1)} t^{\alpha_1 - \beta_1 + \gamma_1 - 1/q} \left(\int_0^t \left[f_1(s) \left| {}^{C}D_0^{\beta_1} y_1(s) \right| + f_2(s) \left| {}^{C}D_0^{\beta_2} y_2(s) \right| \right]^q ds \right)^{\frac{1}{q}}$$

$$\leq K_1 \left(\int_0^t \left[f_1^q(s) \left| {}^{C}D_0^{\beta_1} y_1(s) \right|^q + f_2^q(s) \left| {}^{C}D_0^{\beta_2} y_2(s) \right|^q \right] ds \right)^{\frac{1}{q}}, \; t > 0, \quad (7.24)$$

where we have used Lemma 2.4.4.

Similarly, we see that

$$\left|{}^{C}D_{0}^{\beta_2}y_2\left(t\right)\right| \le K_2\left(\int_0^t\left[f_3^q\left(s\right)\left|{}^{C}D_{0}^{\beta_1}y_1\left(s\right)\right|^q+f_4^q\left(s\right)\left|{}^{C}D_{0}^{\beta_2}y_2\left(s\right)\right|^q\right]ds\right)^{\frac{1}{q}},\ t>0. \tag{7.25}$$

The relation (7.20) is an immediate consequence of (7.21), (7.22), (7.24) and (7.25). ▮

Theorem 7.2.1 *Under the same hypotheses as in Lemma 7.2.1, there exist two positive constants c_i, $i = 1, 2$, such that the solution $(y_1, y_2) \in AC[0,\infty) \times AC[0,\infty)$ of Problem (7.16)-(7.17) satisfies*

$$\left|{}^{C}D_{0}^{\beta_i}y_i\left(t\right)\right| < c_i,\ i=1,2,\ t>0.$$

Proof. By virtue of Lemma 7.2.1, we have

$$z_1\left(t\right) \le K_1\left(\int_0^t\left[f_1^q\left(s\right)z_1^q\left(s\right)+f_2^q\left(s\right)z_2^q\left(s\right)\right]ds\right)^{\frac{1}{q}},\ t>0, \tag{7.26}$$

and

$$z_2\left(t\right) \le K_2\left(\int_0^t\left[f_3^q\left(s\right)z_1^q\left(s\right)+f_4^q\left(s\right)z_2^q\left(s\right)\right]ds\right)^{\frac{1}{q}},\ t>0. \tag{7.27}$$

Raising both sides of (7.26) and (7.27) to the power q, we obtain

$$z_1^q\left(t\right) \le K_1^q\int_0^t\left[f_1^q\left(s\right)z_1^q\left(s\right)+f_2^q\left(s\right)z_2^q\left(s\right)\right]ds,\ t>0,$$

and

$$z_2^q(t) \leq K_2^q \int_0^t \left[f_3^q(s) z_1^q(s) + f_4^q(s) z_2^q(s)\right] ds, \ t > 0.$$

In view of Theorem 6.1.1 and the fact that $f_i \in L_q(0, \infty)$, $i = 1, 2, 3, 4$, we get

$$z_i^q(t) \leq m_i, \ i = 1, 2, \ t > 0,$$

for some positive constants m_i, $i = 1, 2$. Therefore, the result follows from Lemma 7.2.1 with $c_i = m_i^{1/q}$, $i = 1, 2$. ∎

CHAPTER 8

NON-EXISTENCE OF SOLUTIONS FOR A SYSTEM OF NONLINEAR FRACTIONAL DIFFERENTIAL PROBLEMS

In this chapter, we establish necessary conditions for the existence of global solutions to a class of 2×2-systems of fractional differential equations. Namely, we consider the following nonlinear system of fractional differential equations

$$\begin{cases} D_0^{\alpha_1} u_1(t) = f_1\left[t, D_0^{\beta_1} u_1(t), D_0^{\beta_2} u_2(t)\right], \\ D_0^{\alpha_2} u_2(t) = f_2\left[t, D_0^{\beta_1} u_1(t), D_0^{\beta_2} u_2(t)\right], \end{cases} \quad 0 < \beta_1, \beta_2 < \min\{\alpha_1, \alpha_2\} < 1, \ t > 0 \tag{8.1}$$

where D_0^{σ} is either the Riemann-Liouville derivative or the Caputo derivative.

We show that fractional derivatives of lower order have a strong influence on the character of the solutions. Our method of proof relies on a suitable choice of a test function.

8.1 Non-existence of solutions in case of Riemann-Liouville fractional derivatives

In this section, we consider the system (8.1) with Riemann-Liouville fractional derivatives. A nonexistence result of non-trivial global solutions for the system (8.1) will be proved when

$$f_1\left[t, D_0^{\beta_1} u_1(t), D_0^{\beta_2} u_2(t)\right] = t^{\gamma_2} |u_2(t)|^{p_2} - \lambda_1 D_0^{\beta_1} u_1(t), \ t > 0,$$

and

$$f_2\left[t, D_0^{\beta_1} u_1(t), D_0^{\beta_2} u_2(t)\right] = t^{\gamma_1} |u_1(t)|^{p_1} - \lambda_2 D_0^{\beta_2} u_2(t), \ t > 0,$$

for some $p_i > 1$ and $\lambda_i, \gamma_i \in \mathbb{R}$, $i = 1, 2$, in the space $C^{\alpha_1}_{1-\alpha_1}[0, \infty) \times C^{\alpha_2}_{1-\alpha_2}[0, \infty)$, where $C^{\alpha}_{1-\alpha}[0, \infty)$ is as in (4.16). That is, we consider the system

$$\begin{cases} D_0^{\alpha_1} u_1(t) + \lambda_1 D_0^{\beta_1} u_1(t) = t^{\gamma_2} |u_2(t)|^{p_2}, \ 0 < \beta_1 < \alpha_1 < 1, \, t > 0, \\ D_0^{\alpha_2} u_2(t) + \lambda_2 D_0^{\beta_2} u_2(t) = t^{\gamma_1} |u_1(t)|^{p_1}, \ 0 < \beta_2 < \alpha_2 < 1, \, t > 0, \end{cases} \tag{8.2}$$

subject to the initial conditions

$$I_0^{1-\alpha_i} u_i(t) \,|_{t=0} = b_i, \ b_i \in \mathbb{R}, \, i = 1, 2. \tag{8.3}$$

Theorem 8.1.1 *Assume that*

$$1 - \frac{1}{p_1 p_2} < \beta_1 + \frac{\beta_2}{p_1} + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2} \ \text{or} \ 1 - \frac{1}{p_1 p_2} < \beta_2 + \frac{\beta_1}{p_2} + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2}. \tag{8.4}$$

Then, Problem (8.2)-(8.3) admits no non-trivial global solutions in $C^{\alpha_1}_{1-\alpha_1}[0, \infty) \times C^{\alpha_2}_{1-\alpha_2}[0, \infty)$, *when* $b_i \geq 0$, $i = 1, 2$.

Proof. Our proof is by contradiction. Let (u_1, u_2) be a non-trivial global solution and let $\varphi(t)$ be as in (5.2). Multiplying (8.2) by $\varphi(t)$ and integrating over $(0, T)$, we get

$$I_2 := \int_0^T t^{\gamma_2} |u_2(t)|^{p_2} \varphi(t)\, dt = \int_0^T D_0^{\alpha_1} u_1(t) \varphi(t)\, dt + \lambda_1 \int_0^T D_0^{\beta_1} u_1(t) \varphi(t)\, dt, \tag{8.5}$$

and

$$I_1 := \int_0^T t^{\gamma_1} \left| u_1 (t) \right|^{p_1} \varphi (t) \, dt = \int_0^T D_0^{\alpha_2} u_2 (t) \, \varphi (t) \, dt + \lambda_2 \int_0^T D_0^{\beta_2} u_2 (t) \, \varphi (t) \, dt. \tag{8.6}$$

Let

$$J_i = \int_0^T D_0^{\alpha_i} u_i (t) \, \varphi (t) \, dt, \quad i = 1, 2, \tag{8.7}$$

and

$$H_i = \lambda_i \int_0^T D_0^{\beta_i} u_i (t) \, \varphi (t) \, dt, \quad i = 1, 2. \tag{8.8}$$

Next, we shall estimate J_i and H_i, $i = 1, 2$. From the definition of $D_0^{\alpha_i} u_i (t)$ we can write

$$J_i = \int_0^T \varphi (t) \, D_0^{\alpha_i} u_i (t) \, dt = \int_0^T \varphi (t) \, \frac{d}{dt} I_0^{1-\alpha_i} u_i (t) \, dt, \ i = 1, 2, \tag{8.9}$$

and an integration by parts in (8.9) yields

$$J_i = \left[\varphi (t) \, I_0^{1-\alpha_i} u_i (t) \right]_{t=0}^T - \int_0^T \varphi' (t) \, I_0^{1-\alpha_i} u_i (t) \, dt, \ i = 1, 2.$$

Therefore

$$J_i = -b_i - \int_0^T \varphi' (t) \, I_0^{1-\alpha_i} u_i (t) \, dt, \ i = 1, 2.$$

As $b_i \geq 0$, $i = 1, 2$, we see that

$$J_i \leq - \int_0^T \varphi' (t) \, I_0^{1-\alpha_i} u_i (t) \, dt \leq \int_0^T \left| \varphi' (t) \right| \left(I_0^{1-\alpha_i} \left| u_i \right| \right) (t) \, dt$$

$$\leq \frac{1}{\Gamma(1-\alpha_i)} \int_0^T |\varphi'(t)| \int_0^t \frac{|u_i(s)|}{(t-s)^{\alpha_i}} ds dt,\ i=1,2. \tag{8.10}$$

Because $\varphi(t)$ is nonincreasing $\varphi(s) \geq \varphi(t)$ for all $t \geq s$, and therefore

$$\frac{1}{\varphi(s)^{1/p_i}} \leq \frac{1}{\varphi(t)^{1/p_i}},\ p_i > 1,\ i=1,2.$$

Also we have

$$\varphi'(t) = 0,\ t \in [0, T/2].$$

We deduce from (8.10) that

$$J_i \leq \frac{1}{\Gamma(1-\alpha_i)} \int_0^T |\varphi'(t)| \int_0^t \frac{|u_i(s)|}{(t-s)^{\alpha_i}} \frac{\varphi(s)^{1/p_i}}{\varphi(s)^{1/p_i}} ds dt$$

$$\leq \frac{1}{\Gamma(1-\alpha_i)} \int_0^T \frac{|\varphi'(t)|}{\varphi(t)^{1/p_i}} \int_0^t \frac{|u_i(s)|}{(t-s)^{\alpha_i}} \varphi(s)^{1/p_i} ds dt$$

$$\leq \frac{1}{\Gamma(1-\alpha_i)} \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/p_i}} \int_0^t \frac{|u_i(s)|}{(t-s)^{\alpha_i}} \varphi(s)^{1/p_i} ds dt$$

$$\leq \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/p_i}} \left(I_0^{1-\alpha_i} \varphi^{1/p_i} |u_i|\right)(t)\, dt,\ i=1,2. \tag{8.11}$$

By fractional integration by parts (Lemma 2.4.1), we obtain

$$J_i \leq \int_{T/2}^T \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{1/p_i}}\right)(t)\, \varphi(t)^{1/p_i} |u_i(t)|\, dt,\ i=1,2. \tag{8.12}$$

Next, we multiply by $t^{\gamma_i/p_i} t^{-\gamma_i/p_i}$, $i=1,2$, inside the integral in the right hand side of (8.12), we find

$$J_i \leq \int_{T/2}^{T} \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{1/p_i}} \right)(t)\, \varphi(t)^{1/p_i} \frac{t^{\gamma_i/p_i}}{t^{\gamma_i/p_i}} |u_i(t)|\, dt,\ i = 1, 2. \tag{8.13}$$

For $\gamma_i < 0$ we have $t^{-\gamma_i/p_i} < T^{-\gamma_i/p_i}$ (because $t < T$) and for $\gamma_i > 0$ we get $t^{-\gamma_i/p_i} < 2^{\gamma_i/p_i} T^{-\gamma_i/p_i}$ (because $T/2 < t$) : that is

$$t^{-\gamma_i/p_i} < \max\left\{1, 2^{\gamma_i/p_i}\right\} T^{-\gamma_i/p_i},\ i = 1, 2.$$

Then

$$J_i \leq \max\left\{1, 2^{\gamma_i/p_i}\right\} T^{-\gamma_i/p_i} \int_{T/2}^{T} \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{1/p_i}} \right)(t)\, t^{\gamma_i/p_i} \varphi(t)^{1/p_i} |u_i(t)|\, dt,\ i = 1, 2. \tag{8.14}$$

Using Hölder inequality with p_i and $p_i^{'}$ such that $p_i + p_i' = p_i p_i'$, $i = 1, 2$, we find

$$J_i \leq \max\left\{1, 2^{\gamma_i/p_i}\right\} T^{-\gamma_i/p_i} \left(\int_{T/2}^{T} t^{\gamma_i} \varphi(t) |u_i(t)|^{p_i} dt \right)^{\frac{1}{p_i}} \left(\int_{T/2}^{T} \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{1/p_i}} \right)^{p_i'} dt \right)^{\frac{1}{p_i'}}$$

$$\leq \max\left\{1, 2^{\frac{\gamma_i}{p_i}}\right\} T^{\frac{-\gamma_i}{p_i}} \left(\int_{0}^{T} t^{\gamma_i} \varphi(t) |u_i(t)|^{p_i} dt \right)^{\frac{1}{p_i}} \left(\int_{\frac{T}{2}}^{T} \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{\frac{1}{p_i}}} \right)^{p_i'} dt \right)^{\frac{1}{p_i'}},\ i = 1, 2. \tag{8.15}$$

In view of Definitions (8.5) and (8.6), we can write

$$J_i \leq \max\left\{1, 2^{\gamma_i/p_i}\right\} T^{-\gamma_i/p_i} (I_i)^{\frac{1}{p_i}} \left(\int_{T/2}^{T} \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{1/p_i}} \right)^{p_i'} dt \right)^{\frac{1}{p_i'}},\ i = 1, 2. \tag{8.16}$$

Let

$$A_i(T) = \int_{T/2}^{T} \left(I_{T-}^{1-\alpha_i} \frac{|\varphi'|}{\varphi^{1/p_i}} \right)^{p_i'} dt, \ i = 1, 2. \tag{8.17}$$

Lemma 5.1.2 implies that

$$A_i(T) \leq K_{\alpha_i, p_i'} T^{1-\alpha_i p_i'}, \ i = 1, 2. \tag{8.18}$$

From (8.16) and (8.18) we obtain

$$J_i \leq K_i T^{\frac{1}{p_1'} - \alpha_i - \frac{\gamma_i}{p_i}} I_i^{\frac{1}{p_i}}, \ i = 1, 2, \tag{8.19}$$

with

$$K_i = \max \left\{ 1, 2^{\gamma_i / p_i} \right\} K_{\alpha_i, p_i'}^{\frac{1}{p_i'}}, \ i = 1, 2.$$

Now, we estimate H_i, $i = 1, 2$. First, by Lemma 5.1.1, we see that

$$I_0^{1-\beta_i} u_i(0) = \lim_{t \to 0} I_0^{1-\beta_i} u_i(t) = 0, \ i = 1, 2,$$

because $u_i \in C_{1-\alpha_i}[0, T]$ and $1 - \alpha_i < 1 - \beta_i$, $i = 1, 2$. An integration by parts in

$$H_i = \lambda_i \int_0^T \varphi(t) D_0^{\beta_i} u_i(t) \, dt = \lambda_i \int_0^T \varphi(t) \frac{d}{dt} I_0^{1-\beta_i} u_i(t) \, dt, \ i = 1, 2, \tag{8.20}$$

yields

$$H_i = \lambda_i \left[\varphi(t) I_0^{1-\beta_i} u_i(t) \right]_{t=0}^{T} - \lambda_i \int_0^T \varphi'(t) I_0^{1-\beta_i} u_i(t) \, dt, \ i = 1, 2.$$

As $\varphi(T) = 0$ and $I_0^{1-\beta_i} u_i(0) = 0$, $i = 1, 2$, we infer that

$$H_i = -\lambda_i \int_0^T \varphi'(t)\, I_0^{1-\beta_i} u_i(t)\, dt \leq |\lambda_i| \int_0^T |\varphi'(t)| \left(I_0^{1-\beta_i} |u_i|\right)(t)\, dt$$

$$\leq \frac{|\lambda_i|}{\Gamma(1-\beta_i)} \int_0^T |\varphi'(t)| \int_0^t \frac{|u_i(s)|}{(t-s)^{\beta_i}} ds dt$$

$$\leq |\lambda_i| \int_{T/2}^T \frac{|\varphi'(t)|}{\varphi(t)^{1/p_i}} \left(I_0^{1-\beta_i} \varphi^{1/p_i} |u_i|\right)(t)\, dt,\ \ i = 1, 2.$$

Similarly to J_i, $i = 1, 2$, we obtain

$$H_i \leq K_i' T^{\frac{1}{p_i'} - \beta_i - \frac{\gamma_i}{p_i}} I_i^{\frac{1}{p_i}},\ \ i = 1, 2, \tag{8.21}$$

with

$$K_i' = \max\left\{1, 2^{\gamma_i/p_i}\right\} |\lambda_i|\, K_{\beta_i, p_i'}^{\frac{1}{p_i'}},\ \ i = 1, 2.$$

We use (8.19) and (8.21), to write (8.5) and (8.6) in the form

$$I_2 \leq K_1'' T^{\frac{1}{p_1'} - \frac{\gamma_1}{p_1}} I_1^{\frac{1}{p_1}} \left(T^{-\alpha_1} + T^{-\beta_1}\right), \tag{8.22}$$

and

$$I_1 \leq K_2'' T^{\frac{1}{p_2'} - \frac{\gamma_2}{p_2}} I_2^{\frac{1}{p_2}} \left(T^{-\alpha_2} + T^{-\beta_2}\right), \tag{8.23}$$

with

$$K_i'' = \max\{K_i, K_i'\},\ \ i = 1, 2.$$

Consequently (8.22) and (8.23) become

$$I_2^{1-\frac{1}{p_1p_2}} \le K_1''\left(K_2''\right)^{\frac{1}{p_1}} T^{\frac{1}{p_1'}-\frac{\gamma_1}{p_1}+\frac{1}{p_1p_2'}-\frac{\gamma_2}{p_1p_2}} \left(T^{-\alpha_2}+T^{-\beta_2}\right)^{\frac{1}{p_1}} \left(T^{-\alpha_1}+T^{-\beta_1}\right), \quad (8.24)$$

and

$$I_1^{1-\frac{1}{p_1p_2}} \le K_2''\left(K_1''\right)^{\frac{1}{p_2}} T^{\frac{1}{p_2'}-\frac{\gamma_2}{p_2}+\frac{1}{p_2p_1'}-\frac{\gamma_1}{p_2p_1}} \left(T^{-\alpha_1}+T^{-\beta_1}\right)^{\frac{1}{p_2}} \left(T^{-\alpha_2}+T^{-\beta_2}\right). \quad (8.25)$$

Using Lemma 2.4.4 with $0 \le r \le 1$, we estimate the inequalities (8.24) and (8.25) as

$$I_2^{1-\frac{1}{p_1p_2}} \le K_1''\left(K_2''\right)^{\frac{1}{p_1}} T^{\frac{1}{p_1'}-\frac{\gamma_1}{p_1}+\frac{1}{p_1p_2'}-\frac{\gamma_2}{p_1p_2}} \left(T^{-\frac{\alpha_2}{p_1}}+T^{-\frac{\beta_2}{p_1}}\right) \left(T^{-\alpha_1}+T^{-\beta_1}\right),$$

$$I_1^{1-\frac{1}{p_1p_2}} \le K_2''\left(K_1''\right)^{\frac{1}{p_2}} T^{\frac{1}{p_2'}-\frac{\gamma_2}{p_2}+\frac{1}{p_2p_1'}-\frac{\gamma_1}{p_2p_1}} \left(T^{-\frac{\alpha_1}{p_2}}+T^{-\frac{\beta_1}{p_2}}\right) \left(T^{-\alpha_2}+T^{-\beta_2}\right),$$

or

$$I_2^{1-\frac{1}{p_1p_2}} \le K_1''\left(K_2''\right)^{1/p_1}\left(T^{s_1}+T^{s_2}+T^{s_3}+T^{s_4}\right),$$

and

$$I_1^{1-\frac{1}{p_1p_2}} \le K_2''\left(K_1''\right)^{1/p_2}\left(T^{s_5}+T^{s_6}+T^{s_7}+T^{s_8}\right),$$

where

$$s_1 = -\frac{\alpha_2}{p_1} - \alpha_1 + \frac{1}{p_1'} - \frac{\gamma_1}{p_1} + \frac{1}{p_1p_2'} - \frac{\gamma_2}{p_1p_2},$$

$$s_2 = -\frac{\alpha_2}{p_1} - \beta_1 + \frac{1}{p_1'} - \frac{\gamma_1}{p_1} + \frac{1}{p_1p_2'} - \frac{\gamma_2}{p_1p_2},$$

$$s_3 = -\frac{\beta_2}{p_1} - \alpha_1 + \frac{1}{p_1'} - \frac{\gamma_1}{p_1} + \frac{1}{p_1 p_2'} - \frac{\gamma_2}{p_1 p_2},$$

$$s_4 = -\frac{\beta_2}{p_1} - \beta_1 + \frac{1}{p_1'} - \frac{\gamma_1}{p_1} + \frac{1}{p_1 p_2'} - \frac{\gamma_2}{p_1 p_2},$$

$$s_5 = -\frac{\alpha_1}{p_2} - \alpha_2 + \frac{1}{p_2'} - \frac{\gamma_2}{p_2} + \frac{1}{p_2 p_1'} - \frac{\gamma_1}{p_2 p_1},$$

$$s_6 = -\frac{\alpha_1}{p_2} - \beta_2 + \frac{1}{p_2'} - \frac{\gamma_2}{p_2} + \frac{1}{p_2 p_1'} - \frac{\gamma_1}{p_2 p_1},$$

$$s_7 = -\frac{\beta_1}{p_2} - \alpha_2 + \frac{1}{p_2'} - \frac{\gamma_2}{p_2} + \frac{1}{p_2 p_1'} - \frac{\gamma_1}{p_2 p_1},$$

and

$$s_8 = -\frac{\beta_1}{p_2} - \beta_2 + \frac{1}{p_2'} - \frac{\gamma_2}{p_2} + \frac{1}{p_2 p_1'} - \frac{\gamma_1}{p_2 p_1}.$$

If $1 - \frac{1}{p_1 p_2} < \beta_1 + \frac{\beta_2}{p_1} + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2}$, then $s_i < 0$, $i = 1, 2, 3, 4$, or if $1 - \frac{1}{p_1 p_2} < \beta_2 + \frac{\beta_1}{p_2} + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2}$, then $s_j < 0$, $j = 5, 6, 7, 8$, and consequently $T^{s_i} \to 0$ as $T \to \infty$. Thus

$$\lim_{T\to\infty} \int_0^T t^{\gamma_i} \varphi(t) \left| u_i(t) \right|^{p_i} dt = 0, \ i = 1, 2.$$

This implies that $u_i = 0$, $i = 1, 2$. We arrive at a contradiction. ∎

8.2 Non-existence of solutions for a system involving Caputo fractional derivatives

In this section, we consider the system

$$\begin{cases} {}^{C}D_{0}^{\alpha_1} y_1(t) + {}^{C}D_{0}^{\beta_1} y_1(t) \geq t^{\gamma_2} |y_2(t)|^{p_2}, \; 0<\beta_1<\alpha_1<1, \, t>0, \\ {}^{C}D_{0}^{\alpha_2} y_2(t) + {}^{C}D_{0}^{\beta_2} y_2(t) \geq t^{\gamma_1} |y_1(t)|^{p_1}, \; 0<\beta_2<\alpha_2<1, \, t>0, \end{cases} \tag{8.26}$$

with initial conditions

$$y_i(0) = b_i, \, b_i \in \mathbb{R}, \, i = 1,2, \tag{8.27}$$

where ${}^{C}D_{0}^{\sigma}$ is the Caputo fractional derivative, $p_i > 1$ and $\gamma_i \in \mathbb{R}$, $i = 1,2$.

Theorem 8.2.1 *Assume that*

$$1 - \frac{1}{p_1 p_2} < \frac{\beta_2}{p_1} + \beta_1 + \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_1 p_2} \; \text{or} \; 1 - \frac{1}{p_1 p_2} < \frac{\beta_1}{p_2} + \beta_2 + \frac{\gamma_2}{p_2} + \frac{\gamma_1}{p_1 p_2}.$$

Then, Problem (8.26)-(8.27) does not admit nontrivial global solutions in the space $C^{\alpha_1}[0,\infty) \times C^{\alpha_2}[0,\infty)$, where $C^{\alpha}[0,\infty)$ is as in (4.33), when $b_i \geq 0$, $i = 1,2$.

Proof. Assume, on the contrary, that a nontrivial global solution (y_1, y_2) exists for all time $t > 0$. Let φ be as in (5.3) with $\lambda > \frac{p_i \alpha_i}{p_i - 1} - 1$, $i = 1,2$. Multiplying both sides of (8.26) by φ and integrating over $[0,T]$, we get

$$\int_0^T \varphi(t) t^{\gamma_2} |y_2(t)|^{p_2} dt \leq \int_0^T \varphi(t) {}^{C}D_{0}^{\alpha_1} y_1(t) dt + \int_0^T \varphi(t) {}^{C}D_{0}^{\beta_1} y_1(t) dt, \tag{8.28}$$

$$\int_0^T \varphi(t)\, t^{\gamma_1} |y_1(t)|^{p_1}\, dt \leq \int_0^T \varphi(t)^C D_0^{\alpha_2} y_2(t)\, dt + \int_0^T \varphi(t)^C D_0^{\beta_2} y_2(t)\, dt. \quad (8.29)$$

Put

$$I_i = \int_0^T \varphi(t)\, t^{\gamma_i} |y_i(t)|^{p_i}\, dt,\ i = 1, 2, \quad (8.30)$$

$$J_i = \int_0^T \varphi(t)^C D_0^{\alpha_i} y_i(t)\, dt,\ i = 1, 2, \quad (8.31)$$

and

$$H_i = \int_0^T \varphi(t)^C D_0^{\beta_i} y_i(t)\, dt,\ i = 1, 2. \quad (8.32)$$

By Lemma 5.1.3, we have

$$J_i = \int_0^T \varphi(t)^C D_0^{\alpha_i} y_i(t)\, dt = \int_0^T y_i(t)\, D_T^{\alpha_i} \varphi(t)\, dt - \frac{b_i \Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha_i + 2)} T^{1-\alpha_i},\ i = 1, 2, \quad (8.33)$$

and

$$H_i = \int_0^T \varphi(t)^C D_0^{\beta_i} y_i(t)\, dt = \int_0^T y_i(t)\, D_T^{\beta_i} \varphi(t)\, dt - \frac{b_i \Gamma(\lambda + 1)}{\Gamma(\lambda - \beta_i + 2)} T^{1-\beta_i},\ i = 1, 2. \quad (8.34)$$

As $b_i \geq 0$, $i = 1, 2$, we obtain from (8.33) and (8.34)

$$J_i \leq \int_0^T y_i(t)\, D_T^{\alpha_i} \varphi(t)\, dt,\ i = 1, 2, \quad (8.35)$$

and

$$H_i \leq \int_0^T y_i(t)\, D_T^{\beta_i} \varphi(t)\, dt,\ i = 1, 2. \quad (8.36)$$

Next, we multiply by $\varphi^{1/p_i} t^{\gamma_i/p_i} \varphi^{-1/p_i} t^{-\gamma_i/p_i}$, $i = 1, 2$, inside the integrals in the right hand sides of (8.35) and (8.36) respectively, to get

$$J_i \leq \int_0^T y_i(t)\,\varphi^{1/p_i} t^{\gamma_i/p_i} \varphi^{-1/p_i} t^{-\gamma_i/p_i} D_T^{\alpha_i} \varphi(t)\,dt,\ i=1,2, \tag{8.37}$$

$$H_i \leq \int_0^T y_i(t)\,\varphi^{1/p_i} t^{\gamma_i/p_i} \varphi^{-1/p_i} t^{-\gamma_i/p_i} D_T^{\beta_i} \varphi(t)\,dt,\ i=1,2. \tag{8.38}$$

Applying Hölder's inequality with p_i and p_i' such that $p_i + p_i' = p_i p_i'$, $i = 1, 2$, on the integrals in the right hand sides of (8.37) and (8.38), we obtain

$$\begin{aligned} J_i &\leq \left(\int_0^T \varphi(t)\, t^{\gamma_i} |y_i(t)|^{p_i}\,dt\right)^{1/p_i} \left[\int_0^T \varphi^{-p_i'/p_i} t^{-\gamma_i p_i'/p_i} \left|D_T^{\alpha_i}\varphi(t)\right|^{p_i'} dt\right]^{1/p_i'} \\ &\leq I_i^{1/p_i} \left[\int_0^T \varphi^{-p_i'/p_i} t^{-\gamma_i p_i'/p_i} \left|D_T^{\alpha_i}\varphi(t)\right|^{p_i'} dt\right]^{1/p_i'},\ i=1,2, \end{aligned}$$

and

$$\begin{aligned} H_i &\leq \left(\int_0^T \varphi(t)\, t^{\gamma_i} |y_i(t)|^{p_i}\,dt\right)^{1/p_i} \left[\int_0^T \varphi^{-p_i'/p_i} t^{-\gamma_i p_i'/p_i} \left|D_T^{\beta_i}\varphi(t)\right|^{p_i'} dt\right]^{1/p_i'} \\ &\leq I_i^{1/p_i} \left[\int_0^T \varphi^{-p_i'/p_i} t^{-\gamma_i p_i'/p_i} \left|D_T^{\beta_i}\varphi(t)\right|^{p_i'} dt\right]^{1/p_i'},\ i=1,2. \end{aligned}$$

Now, by Lemma 5.1.6, we find

$$J_i \leq I_i^{1/p_i} \left[C_{\lambda,\alpha_i}^{\gamma_i,p_i'} T^{\gamma_i\left(1-p_i'\right)-\alpha_i p_i'+1}\right]^{1/p_i'},\ i=1,2, \tag{8.39}$$

$$H_i \leq I_i^{1/p_i} \left[C_{\lambda,\beta_i}^{\gamma_i,p_i'} T^{\gamma_i\left(1-p_i'\right)-\beta_i p_i'+1}\right]^{1/p_i'},\ i=1,2. \tag{8.40}$$

From (8.28), (8.29), (8.39) and (8.40) we have

$$I_2 \le I_1^{1/p_1} \left(\left[C_{\lambda,\alpha_1}^{\gamma_1,p_1'} T^{\gamma_1\left(1-p_1'\right)-\alpha_1 p_1'+1} \right]^{1/p_1'} + \left[C_{\lambda,\beta_1}^{\gamma_1,p_1'} T^{\gamma_1\left(1-p_1'\right)-\beta_1 p_1'+1} \right]^{1/p_1'} \right)$$

$$\le C_1 I_1^{1/p_1} T^{\frac{\gamma_1\left(1-p_1'\right)+1}{p_1'}} \left(T^{-\alpha_1} + T^{-\beta_1} \right), \tag{8.41}$$

and

$$I_1 \le I_2^{1/p_2} \left(\left[C_{\lambda,\alpha_2}^{\gamma_2,p_2'} T^{\gamma_2\left(1-p_2'\right)-\alpha_2 p_2'+1} \right]^{1/p_2'} + \left[C_{\lambda,\beta_2}^{\gamma_2,p_2'} T^{\gamma_2\left(1-p_2'\right)-\beta_2 p_2'+1} \right]^{1/p_2'} \right)$$

$$\le C_2 I_2^{1/p_2} T^{\frac{\gamma_2\left(1-p_2'\right)+1}{p_2'}} \left(T^{-\alpha_2} + T^{-\beta_2} \right), \tag{8.42}$$

where

$$C_i = \left(\max \left\{ C_{\lambda,\alpha_i}^{\gamma_i,p_i'}, C_{\lambda,\beta_i}^{\gamma_i,p_i'} \right\} \right)^{1/p_i'}, \ i = 1, 2.$$

Therefore

$$I_2^{1-\frac{1}{p_1 p_2}} \le C_1 C_2^{\frac{1}{p_1}} T^{\frac{\gamma_2\left(1-p_2'\right)+1}{p_1 p_2'} + \frac{\gamma_1\left(1-p_1'\right)+1}{p_1'}} \left(T^{-\alpha_2} + T^{-\beta_2} \right)^{\frac{1}{p_1}} \left(T^{-\alpha_1} + T^{-\beta_1} \right)$$

$$\le C_1 C_2^{\frac{1}{p_1}} T^{\frac{\gamma_2\left(1-p_2'\right)+1}{p_1 p_2'} + \frac{\gamma_1\left(1-p_1'\right)+1}{p_1'}} \left(T^{-\alpha_2/p_1} + T^{-\beta_2/p_1} \right) \left(T^{-\alpha_1} + T^{-\beta_1} \right)$$

$$\le C_1 C_2^{\frac{1}{p_1}} \left(T^{s_1} + T^{s_2} + T^{s_3} + T^{s_4} \right), \tag{8.43}$$

and

$$I_1^{1-\frac{1}{p_1 p_2}} \le C_2 C_1^{\frac{1}{p_2}} T^{\frac{\gamma_1\left(1-p_1'\right)+1}{p_2 p_1'} + \frac{\gamma_2\left(1-p_2'\right)+1}{p_2'}} \left(T^{-\alpha_1} + T^{-\beta_1} \right)^{1/p_2} \left(T^{-\alpha_2} + T^{-\beta_2} \right)$$

$$\leq C_2 C_1^{\frac{1}{p_2}} T^{\frac{\gamma_1(1-p_1')+1}{p_2 p_1'}+\frac{\gamma_2(1-p_2')+1}{p_2'}} \left(T^{-\alpha_1/p_2}+T^{-\beta_1/p_2}\right)\left(T^{-\alpha_2}+T^{-\beta_2}\right)$$

$$\leq C_2 C_1^{\frac{1}{p_2}} \left(T^{s_5}+T^{s_6}+T^{s_7}+T^{s_8}\right), \tag{8.44}$$

where we have used Lemma 2.4.4 with $0 \leq r \leq 1$ and

$$s_1 = -\frac{\alpha_2}{p_1} - \alpha_1 + \frac{\gamma_2\left(1-p_2'\right)+1}{p_1 p_2'} + \frac{\gamma_1\left(1-p_1'\right)+1}{p_1'},$$

$$s_2 = -\frac{\alpha_2}{p_1} - \beta_1 + \frac{\gamma_2\left(1-p_2'\right)+1}{p_1 p_2'} + \frac{\gamma_1\left(1-p_1'\right)+1}{p_1'},$$

$$s_3 = -\frac{\beta_2}{p_1} - \alpha_1 + \frac{\gamma_2\left(1-p_2'\right)+1}{p_1 p_2'} + \frac{\gamma_1\left(1-p_1'\right)+1}{p_1'},$$

$$s_4 = -\frac{\beta_2}{p_1} - \beta_1 + \frac{\gamma_2\left(1-p_2'\right)+1}{p_1 p_2'} + \frac{\gamma_1\left(1-p_1'\right)+1}{p_1'},$$

$$s_5 = -\frac{\alpha_1}{p_2} - \alpha_2 + \frac{\gamma_1\left(1-p_1'\right)+1}{p_2 p_1'} + \frac{\gamma_2\left(1-p_2'\right)+1}{p_2'},$$

$$s_6 = -\frac{\alpha_1}{p_2} - \beta_2 + \frac{\gamma_1\left(1-p_1'\right)+1}{p_2 p_1'} + \frac{\gamma_2\left(1-p_2'\right)+1}{p_2'},$$

$$s_7 = -\frac{\beta_1}{p_2} - \alpha_2 + \frac{\gamma_1\left(1-p_1'\right)+1}{p_2 p_1'} + \frac{\gamma_2\left(1-p_2'\right)+1}{p_2'},$$

and

$$s_8 = -\frac{\beta_1}{p_2} - \beta_2 + \frac{\gamma_1\left(1-p_1'\right)+1}{p_2 p_1'} + \frac{\gamma_2\left(1-p_2'\right)+1}{p_2'}.$$

If $1-\frac{1}{p_1 p_2} < \frac{\beta_2}{p_1}+\beta_1+\frac{\gamma_1}{p_1}+\frac{\gamma_2}{p_1 p_2}$, then $s_i < 0$, and consequently $T^{s_i} \to 0$, $i = 1,2,3,4$, as $T \to \infty$.

Similarly, $s_i < 0$, $i = 5,6,7,8$, if $1-\frac{1}{p_1 p_2} < \frac{\beta_1}{p_2}+\beta_2+\frac{\gamma_2}{p_2}+\frac{\gamma_1}{p_1 p_2}$ and consequently

$T^{s_i} \to 0$, $i = 5, 6, 7, 8$, as $T \to \infty$. Thus

$$\lim_{T\to\infty} \int_0^T t^{\gamma_i} \varphi(t) |u_i(t)|^{p_i} \, dt = 0, \ \ i = 1, 2.$$

This implies that $u_i = 0$, $i = 1, 2$. We arrive at a contradiction. ∎

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